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NUMBER THEORY

Lambert, G. Sur des nombres curieux. Mathesis **54**, 148–151 (1940). [MF 5156]

Marcus, O. Distribution des nombres naturels en progressions géométriques n'ayant aucun terme commun. Applications. C. R. Acad. Sci. Paris **211**, 97–99 (1940). [MF 5338]

It is noted that every positive integer can be represented uniquely in the form b^k, where b and k are integers and b is not a power of an integer. Using this principle, one can rearrange certain series, etc. *H. S. Zuckerman.*

Richard, Ubaldo. Risoluzione elementare dell'equazione indeterminata $u^2+v^2=p$, essendo p un numero primo. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 268–273 (1940). [MF 4778]

Well-known proof for the representation of the primes of form $4n+1$ as sums of two squares. *A. Brauer.*

Bell, E. T. Transformed multiplicative diophantine equations. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **8**, 1–21 (1940). (English. Russian summary) [MF 5309]

Let $P_i(x_1, x_2, \cdots, x_n)$ $(i=1, \cdots, m)$ be m polynomials with rational integral coefficients in the n indeterminates $x_1, x_2, \cdots, x_n$. Suppose, further, that the system (1) $P=0$ has the known complete solution (2) $x_j=Q_{ij}(t_1, t_2, \cdots, t_r)$ $(j=1, \cdots, n)$, where $t_1, t_2, \cdots, t_r$ are integral parameters. Let (3) $P(x_1, x_2, \cdots, x_n)=0$ be a given diophantine equation which can be written in the form (4) $P_1 \cdot P_2 \cdots P_m=0$. Then the complete solution of (4) is called the disjunction of the complete solutions of (1) and is the set of m independent solutions (2) for $i=1, 2, \cdots, m$. If the equation (3) can be written in the form $P_1^{s_1}+P_2^{s_2}+\cdots+P_m^{s_m}=0$ ($s_i>0$ and even), then the complete solution of (3) is the $[s_1, s_2, \cdots, s_m]$ conjunction of the complete solutions (2) and is the complete solution of the system (1) considered as a simultaneous system in $x_1, x_2, \cdots, x_n$. The author shows in this paper that the solutions of many of the classical diophantine equations and various new equations are conjunctions and disjunctions of linear or non-linear transforms of equations or systems of equations involving the indeterminates in a multiplicative form. The latter may be solved by the method of reciprocal arrays considered by the author in a previous paper [Amer. J. Math. **55**, 50–66 (1933)]. *I. A. Barnett.*

McKeon, R. P. and Goldstine, H. H. A generalized Pell equation. I. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **8**, 165–172 (1940). (English. Russian summary) [MF 5314]

The authors consider the Pellian cubic

$$x^3+Dy^3+D^2z^3-3Dxzy=1 \tag{1}$$

in the case $D=2$. The paper is a partial rediscovery of the results of Meissel and Mathews, for a general D [Dickson, History of the Theory of Numbers, v. 2, p. 594], of which the authors are apparently unaware. A second paper is promised dealing with the general equation (1). *D. H. Lehmer* (Berkeley, Calif.).

Skolem, Th. On orthogonally situated lattice-points on spheres. Norsk. Mat. Tidsskr. **23**, 54–61 (1941). (Norwegian) [MF 4940]

Consider a sphere Σ of radius r about O. Two points P_i of Σ are called orthogonal lattice points if their coordinates are integers and if the vectors OP_i are orthogonal. For the existence of a mutually orthogonal triple it is necessary and sufficient that r be an integer. If r is odd, then any lattice point P on Σ belongs to an orthogonal triple. Among the other results of the paper we mention: Σ contains two orthogonal lattice points if and only if the Diophantine equation $u^2+v^2=n$, where $n=r^2$, has a solution (here r is not necessarily an integer). *W. Feller* (Providence, R. I.).

Gupta, Hansraj. Some properties of generalised combinatory functions. J. Indian Math. Soc. (N.S.) **5**, 27–31 (1941). [MF 5110]

The author defines the symbol $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_a$ as the sum of products of the numbers $1, a, a^2, \cdots, a^{n-1}$ taken r at a time, and points out their similarities with binomial coefficients, such as $\left[\begin{smallmatrix} n+1 \\ r \end{smallmatrix}\right]_a=\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_a+a^n\left[\begin{smallmatrix} n \\ r-1 \end{smallmatrix}\right]_a$ and $\sum_{r=0}^{n}(-1)^r\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_a=0$, both of which were known to Gauss. He next gives two theorems giving congruence properties of these generalized binomial coefficients for very special values of n and a. *D. H. Lehmer* (Berkeley, Calif.).

Gupta, Hansraj. Some idiosyncratic numbers of Ramanujan. Proc. Indian Acad. Sci., Sect. A. **13**, 519–520 (1941). [MF 5244]

The author lists all odd numbers not greater than 20000 not of the form $x^2+y^2+10z^2$ as follows: 3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719. All but the last two numbers were given by Ramanujan, who remarked that these numbers obey no obvious law. There is also given a table of the number $N_3(x)$ of non-negative integers not greater than x which are sums of two non-negative squares, for $x=4, 100$ and $1000k$, $k=1, 2, 3, \cdots, 20$. These values are compared with an empirical formula

$$N_3(x)=\frac{.764x}{(\log x)^{\frac{1}{2}}}+\frac{.52x^{31/40}}{(\log x)^{\frac{1}{2}}}+212\left\{1-\cos\frac{\pi x}{9.104}\right\},$$

in which the first term is asymptotically correct [see these Rev. **1**, 201]. *D. H. Lehmer* (Berkeley, Calif.).

Kesava Menon, P. A theorem on congruence. Math. Student **8**, 156–158 (1940). [MF 5002]

Previously the author proved the following theorem [J. Indian Math. Soc. (N.S.) **2**, 332–333 (1937)]: Let p be a

prime, then we have

$$(1) \qquad \sum_{n=1}^{p-1} n^{p-1} \equiv p+(p-1)! \pmod{p^2}.$$

It seems not to be noticed that this result is not new. It was obtained by Lerch [Math. Ann. 60, 471–490 (1905)]. In the present paper the author generalizes (1) as follows. Let m be any integer and $n_1, n_2, \cdots, n_{\varphi(m)}$ any reduced system of residues. Let c_r be the rth elementary symmetric function of $n_1^{\varphi(m)}, n_2^{\varphi(m)}, \cdots, n_{\varphi(m)}^{\varphi(m)}$. Then we have

$$c_r \equiv -\binom{\varphi(m)}{r}(r-1 \pm r n_1 n_2 \cdots n_{\varphi(m)}),$$

where the positive sign is to be taken for $m=4$, p^k, $2p^k$ (p odd prime), and the negative sign in any other case. [On page 156, lines 14 and 16 read $n_{\varphi(m)}$ instead of $n^{\varphi(m)}$; on page 158, line 6 read $1 \mp \lambda m^{\varphi} r$ instead of $1 \mp \lambda m^{\varphi} r$.]

A. Brauer (Princeton, N. J.).

Schwarz, Štefan. Sur le nombre des racines et des facteurs irréductibles d'une congruence donnée. Časopis Pěst. Mat. Fys. 69, 128–145 (1940). (French. Czech and German summaries) [MF 5205]

Consider the congruence

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \equiv 0 \pmod{p},$$

where the coefficients are integers, p is a prime and the discriminant D is not divisible by p. Formulas for the number of integral solutions of this congruence have been given by many writers [see Dickson, History of the Theory of Numbers, vol. I.]. In this paper the author proposes and solves the more general problem of determining r_k, the number of irreducible factors mod p of degree k. The results are given in terms of determinants whose elements are various power sums of the n imaginary solutions of the given congruence, which are contained in a suitable extension of the field of residues mod p. For r_1 the theorem reads as follows: Let j_i, $1 \leqq i \leqq n$, denote the imaginary solutions of the congruence $s_k \equiv j_1^k + \cdots + j_n^k \pmod{p}$, $s_{\alpha\beta} = s_{\alpha+\beta}$, $0 \leqq \alpha \leqq n-1$, $0 \leqq \beta \leqq n-1$. Then the determinant $D = |s_{\alpha\beta}|$ is the discriminant of the congruence. If $D_{\alpha\beta}$ is the minor of the element $s_{\alpha\beta}$ of D, we have

$$r_1 \equiv D^{-1} \sum_{\alpha=0}^{n-1} \sum_{\beta=0}^{n-1} s_{p\alpha+\beta} D_{\alpha\beta} \pmod{p}.$$

A similar formula holds in terms of the determinant $d = |t_{\alpha\beta}|$, where $t_{\alpha\beta} = s_{p\alpha+\beta}$, and the minors $d_{\alpha\beta}$ of the elements $t_{\alpha\beta}$ in d. The general result follows the same pattern, but is too complicated to write down in a short space. The paper concludes with applications to quadratic, cubic and binomial congruences, and a proof of the result $(D|p) = (-1)^{n+\nu}$, where ν is the number of irreducible factors mod p of the given congruence of degree n.

R. D. James (Saskatoon, Sask.).

Bunický, E. Über ein System von Kongruenzen, welches mit dem Wilsonschen Satz zusammenhängt. Časopis Pěst. Mat. Fys. 69, 97–109 (1940). (Czech. German summary) [MF 5203]

Segond, Marcel. Sur l'exposant du facteur b dans $A^m - 1$ et sur les développements systématiques des nombres rationnels. C. R. Acad. Sci. Paris 212, 470–472 (1941). [MF 4916]

This paper contains without proofs statements which are in part obvious, in part incorrect. *A. Brauer.*

Sispánov, S. On pseudo-prime numbers. Bol. Mat. 14, 99–106 (1941). (Spanish) [MF 4824]

A number m that is not a prime is called pseudo-prime if $a^{m-1} \equiv 1 \pmod{m}$ for all a with $(a, m) = 1$. A pseudo-prime is necessarily odd, quadratfrei and has at least three prime-factors. The number $m = p_1 \cdots p_k$ ($k \geqq 3$, $2 < p_1 < \cdots < p_k$) is pseudo-prime if and only if $p_i - 1 | m-1$ for $i = 1, \cdots, k$. This leads to the following construction: Let $x_1, \cdots, x_k$ be any integers with $(x_1, \cdots, x_k) = 1$ and let s run through the integers for which

$$\sum x_1 + s \sum x_1 x_2 + \cdots + s^{k-2} \sum x_1 \cdots x_{k-1}$$

is divisible by the smallest common multiple y of the numbers $x_1, \cdots, x_k$. [If the set of the numbers s is not vacuous, it obviously consists of certain classes of remainders (mod y); the sums indicate elementary symmetric functions of $x_1, \cdots, x_k$.] If s can be chosen in such a way that the k numbers $sx_i + 1$ are simultaneously prime numbers, their product is pseudo-prime. All the pseudo-primes can be obtained in this way, and each of them only once.

P. Scherk (Bloomington, Ind.).

Beeger, N. G. W. H. Extension of the table of least exponents ξ for which $2^\xi \equiv 1 \pmod{p}$. Nieuw Arch. Wiskde (2) 20, 307–308 (1940). [MF 5216]

This small table extends from 300,000 to 309,671 Kraitchik's table of the residue-index of 2 modulo p [Recherches sur la Theorie des Nombres, Paris, 1924, v. 1, pp. 131–191]. The largest entry is 702 corresponding to $p = 309{,}583$, thus showing that this prime is a factor of $2^{441} - 1$. *D. H. Lehmer* (Berkeley, Calif.).

Chung, Kai-Lai. Note on a theorem on quadratic residues. Bull. Amer. Math. Soc. 47, 514–516 (1941). [MF 4550]

Short proof of the known theorem: For every prime $p \equiv 3 \bmod 4$ there are more quadratic residues between 0 and $p/2$ then between $p/2$ and p. This is equivalent to

$$\sum u(s, p) = \sum_{s=1}^{(p-1)/2} \{[2s^2/p] - 2[s^2/p]\} \leqq (p-1)/4.$$

Using the sine Fourier series for $x - [x] - \frac{1}{2}$, we have

$$u(s, p) = \sum_n (\sin(4n\pi s^2/p) - 2\sin(2\pi s^2/p))/n\pi,$$

with the first summation running from $n = 1$ to ∞. Making use of Gaussian sums, it is shown easily that

$$\sum_n (1/n\pi) \sum_s \sin(4n\pi s^2/p) \leqq \sum_n (2/n\pi) \sum_s \sin(2n\pi s^2/p).$$

A. J. Kempner (Boulder, Colo.).

Pillai, S. S. On m consecutive integers. III. Proc. Indian Acad. Sci., Sect. A. 13, 530–533 (1941). [MF 5247]

[The first two notes appeared in the same Proc. 11, 6–12, 73–80 (1940); cf. these Rev. 1, 199, 291.] It is proved that, when $m \geqq 17$, there exists a set of consecutive integers such that no number in the set is prime to all the rest. The same theorem has been proved by A. Brauer [Bull. Amer. Math. Soc. 47, 328–331 (1941); these Rev. 2, 248] and the methods of proof are similar. For m sufficiently large the result follows directly from a theorem of P. Erdös [Quart. J. Math. 6, 205–213 (1935)]. The author uses Erdös' methods to prove the result for $m \geqq 12321$ and then takes care of the remaining values of m by means of some short tables. The tables contain two confusing misprints: under $635 \leqq m \leqq 1045$ the entry 571 should be replaced by 563; the heading $3388 \leqq m \leqq 4095$ should read $3338 \leqq m \leqq 4095$.

H. S. Zuckerman (Seattle, Wash.).

Vandiver, H. S. Simple explicit expressions for generalized Bernoulli numbers of the first order. Duke Math. J. **8**, 575–584 (1941). [MF 5194]

The author defines generalized Bernoulli numbers of the first order by the umbral equality $b_n(m, k)=(mb+k)^n$, where $b_n(0, 1)$ is the ordinary Bernoulli number. Explicit expressions for $b_n(m, k)$ in terms of the sum

$$S_n(m, k, a)=\sum_{i=0}^{a-1}(mi+k)^n$$

are given, such as

$$b_{n-1}(m, k)=\sum_{a=1}^{r}(-1)^{a-1}\binom{r}{a}S_{n-1}(m, k, a)/a.$$

Divisibility properties of the $b_n(m, k)$ are also considered, for example, $b_n(m, k)\equiv 0 \pmod{p^a}$, where $n=n_1p^a$, $n_1\not\equiv 0 \pmod{p-1}$ and $(m, p)=1$, p a prime. Another proof is given of the author's generalization of the von Staudt-Clausen theorem [see Proc. Nat. Acad. Sci. U. S. A. **23**, 555–559 (1937)], which is as follows: For n even

$$b_n(m, k)=A_n-\sum_{i=1}^{s}1/p_i,$$

where the p's are distinct primes which are prime to $m\neq 0$ and such that $n\equiv 0 \pmod{p_i-1}$, A_n being some integer. For n odd, $b_n(m, k)$ is an integer, except for $n=1$ with m odd.

D. H. Lehmer (Berkeley, Calif.).

Carlitz, L. Generalized Bernoulli and Euler numbers. Duke Math. J. **8**, 585–589 (1941). [MF 5195]

This paper gives still another proof of the generalization of the von Staudt-Clausen theorem mentioned in the preceding review. The author also considers a generalization of Bernoulli numbers to the case of multiple indices, as follows:

$$b_{n_1\cdots n_t}=\prod_{i=1}^{t}(m_ib+k_i)^{n_i},$$

and proves a von Staudt-Clausen theorem for these numbers. Eulerian numbers are generalized in the same way, and a complicated divisibility property is established for them.

D. H. Lehmer (Berkeley, Calif.).

Hull, Ralph. The representation of integers in forms. Nat. Math. Mag. **14**, 235–252 (1940). [MF 5251]

Expository article.

Benneton, Gaston. Sur la représentation des nombres entiers par une somme de 2^m carrés et sa mise en facteurs. C. R. Acad. Sci. Paris **212**, 591–593, 637–639 (1941). [MF 4920, 4924]

The author considers the representation of numbers as sums of 4 squares. An ordered set of four integers $[a, b, c, d]$ which are not necessarily positive such that $a^2+b^2+c^2+d^2=p$ is called a form of p and is denoted by P. Here p is any integer called the argument of P. If a, b, c and d are replaced by their absolutely least residues modulo m we obtain a new form R of argument less than m^2 called the remainder of P on division by m. The product of two forms $P=[a, b, c, d]$ and $Q=[x, y, z, t]$ is denoted by PQ and is defined as the form

$$PQ=[ax+by+cz+dt,\ -bx+ay-dz+ct,\ -cx+dy+az-bt,\ -dx-cy+bz+at],$$

so that the argument of a product of forms is the product of the arguments of these forms. Euler's simplification of a part of Lagrange's proof of the 4 square theorem is translated into the above theory to give a very simple proof of the fact that, if p is a prime and if pq is the sum of four squares, so also is p. The author proceeds to problems of which the following is the simplest. Let X be any primitive form (that is, one whose elements have no common factor) of argument x, and let e be any integer, to find the number of forms E of e for which the form EX is likewise primitive. The number depends not on X but only on e and x and is given by $8Ke\prod(1+1/p)$, where the product extends over all those primes p dividing e but not x. The constant $K=1, 0$ or $\frac{1}{4}$ in different cases. The problem of the number of decompositions of a given form into a product of k forms is considered also, especially for $k=2$ and 3.

D. H. Lehmer (Berkeley, Calif.).

Olds, C. D. On the representations, $N_7(m^2)$. Bull. Amer. Math. Soc. **47**, 624–628 (1941). [MF 5057]

The author gives a purely arithmetical proof of the theorem that the number $N_7(m^2)$ of representations of $m^2=(p_1^{a_1}p_2^{a_2}\cdots p_s^{a_s})^2$ as the sum of 7 squares is given by

$$N_7(m^2)=14\prod_{r=1}^{s}\{\sigma_3(p_r^{a_r})-(-1)^{(p_r-1)/2}p_r\sigma_3(p_r^{a_r-1})\}.$$

The method is similar to the one used in a recent paper [same vol., 499–503; these Rev. **2**, 348] to obtain $N_3(m^2)$. The scheme is based on the fact that the arguments of f in a typical sum $\sum_x f(n-kx^2)$ become factorable when n and k are squares, a fact first noted and used by Hurwitz in discussing $N_5(m^2)$.

D. H. Lehmer (Berkeley, Calif.).

Niven, Ivan. Sums of n-th powers of quadratic integers. Duke Math. J. **8**, 441–451 (1941). [MF 5181]

The author first determines those quadratic fields $R(\sqrt{m})$ in which every integer is expressible in the form $\alpha_1^n+\alpha_2^n+\cdots+\alpha_N^n$, where α_i are integers in the field. This is quite simple when n is odd, in fact the necessary and sufficient condition is that $(m, n)=1$ and, in case n is a multiple of 3, that $m\not\equiv 5 \pmod 8$. A similar but more complicated condition is later given for n even. Next he determines when a given integer in $R(\sqrt{m})$ can be expressed in this fashion. The condition for this is again simple when n is odd; when n is even only the case of an imaginary field ($m<0$) is treated. In each case it is tacitly understood that N is independent of the given integer and depends only on n and m. The methods used are elementary, although the identity used by Hilbert in his proof of Waring's theorem [Math. Ann. **67**, 283 (1909)] is used at one point.

H. W. Brinkmann (Swarthmore, Pa.).

Wachs, Sylvain. Sur certains aspects analytiques du théorème de Fermat. C. R. Acad. Sci. Paris **211**, 55–57 (1940). [MF 5333]

This note contains three "analytic" theorems that are equivalent to Fermat's last theorem. For example, to prove Fermat's theorem is equivalent to proving that, if a, b, c are integers, then no one of the coefficients in the power series expansion in terms of z of the function

$$\frac{d}{dz}\log(1-az)(1-bz)(1-cz)$$

can be zero.

H. W. Brinkmann (Swarthmore, Pa.).

Bitterlich-Willmann, Johann. Eine Verallgemeinerung der Fermatschen Vermutung. J. Reine Angew. Math. **183**, 251–252 (1941). [MF 5727]

The author conjectures that the number of solutions (in positive integers) of $x^k+y^k=z^n$ with $z \leqq M$ is asymptotic to $C \cdot M^{1+2n/k-n}$. This is true for $(n, k)=(2, 2)$, $(3, 3)$, $(2, 4)$, but it seems difficult to make it plausible in general. If true, it would obviously imply Fermat's famous theorem.

H. W. Brinkmann (Swarthmore, Pa.).

Inaba, Eizi. Klassenkörpertheoretische Deutung der Struktur der Klassengruppe des zyklischen Zahlkörpers. Proc. Imp. Acad. Tokyo **17**, 125–128 (1941). [MF 5392]

Let K be a cyclic number field of prime degree l over the field of rational numbers R. In a previous paper the author has studied the structure of the l-class group H of K, this being a certain subgroup of the group G of ideal classes of K [J. Fac. Sci. Imp. Univ. Tokyo Sect. 1. **4**, 61–115 (1940); these Rev. **2**, 147]. In the present paper the author interprets these results according to the general plan of the class-field theory, by showing how one may construct the fields (Abelian over K) that correspond to the quotient group G/H and its various subgroups. For $l=2$ this had already been done by S. Iyanaga [Actual Sci. Ind., no. 197, Paris, 1935] and H. Reichardt [J. Reine Angew. Math. **170**, 69–74 (1933)]. *H. W. Brinkmann.*

Pillai, S. S. On the sum function connected with primitive roots. Proc. Indian Acad. Sci., Sect. A. **13**, 526–529 (1941). [MF 5246]

Let $P(n)$ denote the number of primitive roots of n, $\phi(x)$ and $\mu(x)$ the functions of Euler and Möbius, respectively; let $S(x)=\sum_{p \leqq x}\phi(p-1)$, $T(x)=\sum_{n \leqq x}P(n)$, $A=\sum_{n=1}^{\infty}\mu(n)/n\phi(n)$, $\mathrm{li}(x)=\int_2^x dt/\log t$. Two theorems are proved in this paper:

(I) $S(x)=A\,\mathrm{li}(x^2)+O\{x^2/(\log x)^m\}$;

(II) $T(x)=A\{\mathrm{li}(x^2)+\mathrm{li}(x^2/4)\}+O\{x^2/(\log x)^m\}$

for any given positive integer m. Theorem (II) follows from theorem (I), which depends on an estimate of $S_d=\sum p$ $(p \leqq x;\ p \equiv 1 \pmod d)$ for $d \leqq (\log x)^m$. This, in turn, depends on an estimate of the number of primes in an arithmetical progression. It is stated that a similar result holds for $\sum_{p \leqq x}\sigma_{-1}(p-1)$, where $\sigma_{-1}(n)=\sum_{d|n}(1/d)$.

R. D. James (Saskatoon, Sask.).

Walfisz, Arnold. Zur additiven Zahlentheorie. VII (1). Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 7–14 (1941). (German. Russian summary) [MF 5292]

Walfisz, Arnold. Zur additiven Zahlentheorie. VII (2). Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 221–226 (1941). (German. Russian summary) [MF 5299]

The two papers deal with Goldbach's problem and its generalization to the representation of an integer as a sum of r primes, where $r \geqq 3$. The method of proof is a refinement of that of Vinogradoff, but it uses the function

$$\nu_r(n)=\sum \log p_1 \cdots \log p_r, \quad p_1+\cdots+p_r=n,$$

instead of the direct number of solutions function

$$N_r(n)=\sum 1, \qquad p_1+\cdots+p_r=n.$$

The first part establishes the result

$$\nu_r(n)=\frac{n^{r-1}}{(r-1)!}S_r(n)+O\left(\frac{n}{\log n}\right)^{r-1},$$

where the constant in the O-symbol is independent of r and $S_r(n)$ is the singular series. In the second part the formula

$$N_r(n)=\frac{n^{r-1}}{(r-1)!\log n}S_r(n)+o\left(\frac{n^{r-1}}{\log^r n}\right),$$

assumed true for $r=3$, is proved in general by induction on r. [Cf. also the following review.]

R. D. James (Saskatoon, Sask.).

Walfisz, Arnold. Zur additiven Zahlentheorie. VIII. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **8**, 69–107 (1940). (German. Russian summary) [MF 5311]

This paper is a continuation of VI in the same series [Trav. Inst. Math. Tbilissi **5**, 197–253 (1938)]. It deals with the representation of integers by certain quaternary quadratic forms, nineteen in all, such as $x^2+y^2+z^2+3t^2$. The number of representations is given in terms of divisor functions. Some of the results have been stated without proof by Liouville, Jacobi and others, but here they are gathered together systematically in one paper.

R. D. James (Saskatoon, Sask.).

Chung, Kai-Lai. Two remarks on Viggo Brun's method. Sci. Rep. Nat. Tsing Hua Univ. (A) **4**, 249–255 (1940). [MF 5326]

Romanoff has proved that the set of integers of the form $p+a^i$, where p is a prime and $i \geqq 0$, has positive density. [See, for example, E. Landau, Über einige neuere Fortschritte der additiven Zahlentheorie, Cambridge Tracts, no. 35, London, 1937, pp. 63–70.] In this paper it is shown that the set of integers $\pi p+a^{p_i}$, where π and p are different primes and the p_i are also primes, has positive density. The method of proof parallels that given by Landau in the above-mentioned reference for the Romanoff result. A generalization of the Viggo Brun result on prime pairs is also indicated. *R. D. James* (Saskatoon, Sask.).

Brauer, Alfred. On the density of the sum of sets of positive integers. II. Ann. of Math. (2) **42**, 959–988 (1941). [MF 5525]

Let $A_1, A_2, \cdots, A_n$ be sets of positive integers with the respective positive densities $\alpha_1, \alpha_2, \cdots, \alpha_n$, that is,

$$\alpha_k=\inf_{n=1,2,\cdots} n^{-1}\sum_{\substack{a_k \leqq n \\ a_k \in A_k}} 1.$$

Let $\gamma<1$ be the density of the set $A_1+A_2+\cdots+A_n$ of all positive integers of the form $a_1+a_2+\cdots+a_n$, where $a_k \in A_k$ or $a_k=0$. The so-called (α, β)-hypothesis reads $\gamma \geqq \alpha_1+\alpha_2$ for $n=2$. From it, $\gamma \geqq \alpha_1+\alpha_2+\cdots+\alpha_n$ would follow. The author continues and refines numerical research by Schur [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **1936**, 269–297] and himself [Ann. of Math. (2) **39**, 322–340 (1938)], where γ is estimated by means of the α's and related problems are discussed. In the present paper, he proves, for example, for $n=2$ that $\gamma \geqq 9(\alpha_1+\alpha_2)/10$ and $\gamma \geqq \alpha_1+5\alpha_2/6$ if $\alpha_1 \leqq \alpha_2$.

P. Scherk (Bloomington, Ind.).

Kantz, Georg. Zerfällung einer Zahl in Summanden. Deutsche Math. **5**, 476–481 (1941). [MF 4799]

The author considers the number $\{{s \atop r}\}_p$ of partitions of the integer s into r parts each greater than or equal to p, and proves a number of simple relations and recursion formulas for this function. Since obviously $\{{s \atop r}\}_p=\{{s-(p-1)r \atop r}\}_1$, it

suffices to find a formula for the case $p=1$. The author derives a formula equivalent to (27) on page 128 of Netto's Lehrbuch der Combinatorik [Leipzig and Berlin, 1927]. Netto's table of this function is extended to $s\leqq 25$, $r\leqq 15$.

D. H. Lehmer (Berkeley, Calif.).

Erdös, Paul and Lehner, Joseph. The distribution of the number of summands in the partitions of a positive integer. Duke Math. J. 8, 335–345 (1941). [MF 4597]

Denote by $p(n)$ the number of unrestricted partitions of a positive integer n and by $p_k(n)$ the number of partitions of n which have at most k summands. The authors prove that if $k=C^{-1}n^{\frac{1}{2}}\log n+xn^{\frac{1}{2}}$ then

$$\lim_{n\to\infty}\frac{p_k(n)}{p(n)}=\exp(-2C^{-1}e^{-\frac{1}{2}Cx}).$$

Here $C=\pi(\frac{2}{3})^{\frac{1}{2}}$. The proof makes use of the sieve method of Brun and the well-known asymptotic formula for $p(n)$. A similar result is given for $P(n)$, the number of partitions of n into unequal parts. In addition the asymptotic formula

$$p_k(n)\sim\binom{n-1}{k-1}/k!$$

valid uniformly in k in the range $k=o(n^{\frac{1}{2}})$ is established without using the calculus of residues. *R. D. James.*

Haberzetle, Mary. On some partition functions. Amer. J. Math. 63, 589–599 (1941). [MF 4683]

The coefficient a_n of x^n in

$$F(x)=\frac{f(x)f(x^{pq})}{f(x^p)f(x^q)}=\sum_{n=0}^{\infty}a_nx^n,$$

where p and q are different primes and

$$f(x)=\prod_{m=1}^{\infty}(1-x^m)^{-1},$$

is the number of partitions of n into parts none of which is a multiple of p or of q. The author applies the reviewer's variant of the Hardy-Ramanujan method to obtain a convergent infinite series for a_n. The resulting expression, which is otherwise of the customary type, depends linearly on those a_ν with (*) $\nu<\mu=(p-1)(q-1)/24$. This is in accordance with the order of the pole at $\tau=i\infty$ of the modular function corresponding to $F(x)$.

The author assumes that μ defined in (*) is an integer, which serves to simplify the roots of unity entering into the Kloosterman sums. However, this condition is unessential in the proof. Thus this paper covers also I. Nivens' result for $p=2$, $q=3$ [Amer. J. Math. 62, 353–364 (1940); cf. these Rev. 1, 201], although the author disclaims it.

H. Rademacher (Philadelphia, Pa.).

Fogels, E. On average values of arithmetic functions. Proc. Cambridge Philos. Soc. 37, 358–372 (1941). [MF 5544]

This paper deals with the problem of finding functions $h=h(x)$ for which the mean relation $\sum_{n=1}^{x}a(n)\sim A\psi(x)$, $x\to\infty$, can be extended to (1) $\sum_{n=x}^{x+h}a(n)\sim A\psi(x+h)-A\psi(x)$, $x\to\infty$. By using a method of Heilbronn [Math. Z. 36, 394–423 (1933)] and a convexity theorem of Ingham [Quart. J. Math. 8, 255–266 (1937), Th. 2], the author proves that (1) holds when $a(n)$ is either of the arithmetical functions $\Lambda(n)$, $\lambda(n)$ or $\mu(n)$ and $h(x)=x^\theta$, $\theta>(1+4c)/(2+4c)$, where c is a constant for which $\zeta(\frac{1}{2}+it)=O(t^c)$, $t\to\infty$. [The result for $\Lambda(n)$ had been proved earlier in another manner by Ingham, loc. cit.] Similar results are obtained for the arithmetical functions $q(n)$, $2^{\nu(n)}$, $d(n^2)$ and $d^2(n)$. *P. Hartman.*

Kac, M. Two number-theoretic remarks. Revista Ci., Lima 43, 177–182 (1941). [MF 4847]

The author proves the following results: (1) Denote by $d(a,h,k)$ the density of integers n which have exactly k prime divisors between a and a^h. Then $\lim_{a\to\infty}d(a,h,k)=e^{-h}h^k/k!$.

$$(2)\qquad \sum_{k=0}^{\infty}(-1)^k\frac{d(2k+1)}{2k+1}=\frac{\pi^2}{16}.$$

This was a problem proposed by Ramanujan. *P. Erdös.*

Kac, M. Note on the distribution of values of the arithmetic function $d(m)$. Bull. Amer. Math. Soc. 47, 815–817 (1941). [MF 5494]

The author proves the following theorem: Denote by $r_n(\omega)$ the number of integers $m\leqq n$ for which

$$d(m)\leqq 2^{\log\log n+\omega(2\log\log n)^{\frac{1}{2}}}$$

($d(m)$ denotes the number of divisors of m). Then

$$\lim_{n\to\infty}r_n(\omega)/n=\pi^{-\frac{1}{2}}\int_{-\infty}^{\omega}e^{-u^2}du.$$

The proof follows from a joint result of the author and reviewer [Amer. J. Math. 62, 738–742 (1940); cf. these Rev. 2, 42]. *P. Erdös* (Philadelphia, Pa.).

Erdös, P., Kac, M., van Kampen, E. R. and Wintner, A. Ramanujan sums and almost periodic functions. Studia Math. 9, 43–53 (1940). (English. Ukrainian summary) [MF 5255]

If $f(n)$ has the property $f(n_1n_2)=f(n_1)f(n_2)$ and $f(p)=f(p^2)=f(p^3)=\cdots$ ($p=$ prime), and if $\sum|f(p)-1|/p$ is finite, then the sequence $\{f(n)\}$ is almost periodic (B) and the familiar Ramanujan expansion

$$c+c\sum_{q>1}\sum_{m}\sum_{p\mid q}\frac{f(p)-1}{f(p)-1+p}\cos\left(2\pi\frac{m}{q}p\right)$$

(with the familiar restrictions on p, q, m) is its formal Fourier expansion. The sequence is almost periodic (B^2) if $\sum|f(p)^2-1|/p$ is also finite and it is uniformly almost periodic if $\sum|f(p)-1|$ is finite. *S. Bochner.*

Wintner, Aurel. On the asymptotic behavior of the Riemann zeta-function on the line $\sigma=1$. Amer. J. Math. 63, 575–580 (1941). [MF 4680]

The author denotes by $\tau_r(T)$ the measure of the set of points t of the interval $-T\leqq t\leqq T$ at which $|\zeta(1+it)|\leqq r$, and defines $\psi(r)$, the asymptotic distribution function of $r=|\zeta(1+it)|$, to be the limit of $\tau_r(T)/(2T)$ as $T\to\infty$. B. Jessen and A. Wintner [Trans. Amer. Math. Soc. 38, 48–88 (1935)] found that $\psi(r)=O\exp(-\lambda\log^2 r)$ as $r\to 0$ for every fixed $\lambda>0$; the author here improves this estimate, and shows that $\psi(r)=O\exp(-\lambda/r)$ as $r\to 0$ (for every fixed $\lambda>0$). The reasoning is based on the author's result that $1/\zeta(1+it)$ is almost periodic (B^2) and

$$(1)\qquad 1/\zeta(1+it)\sim\sum_{n=1}^{\infty}\mu(n)n^{-1}\exp(-it\log n).$$

H. R. Pitt's extension of the Hausdorff-Young theorem

applied to $\sum_{n=m}^{\infty}\mu(n)n^{-1}\exp(-it\log n)=S_m$, say, yields for every positive integral k a function $f_{2k}^{(m)}\sim S_m$, and an inequality for the mean of $|f_{2k}^{(m)}|$, from which (using results of the paper cited above) the author concludes that

$$\left[\int_0^{\infty} r^{2k}d\phi(r)\right]^{1/2k}=o(k) \tag{2}$$

as $k\to\infty$, where ϕ is the asymptotic distribution function of $1/|\zeta(1+it)|$. (2) is equivalent to the above estimation for $\psi(r)$. *D. C. Spencer* (Cambridge, Mass.).

Atkinson, F. V. The mean value of the zeta-function on the critical line. Proc. London Math. Soc. (2) **47**, 174–200 (1941). [MF 5401]

This paper consists of a proof of the asymptotic formula

$$\int_0^{\infty}|\zeta(\tfrac{1}{2}+it)|^4e^{-\delta t}dt=\delta^{-1}(A\log^4 1/\delta+B\log^3 1/\delta+C\log^2 1/\delta+D\log 1/\delta+E)+O((1/\delta)^{13/14+\epsilon}),$$

as $\delta\to 0$, for $\epsilon>0$. Here $A=(2\pi^2)^{-1}$ and B, C, D, E are other constants. The value of B is given in the paper and the other constants could also be found by tracing them through the proof. The formula is an improvement on the formula

$$\int_0^{\infty}|\zeta(\tfrac{1}{2}+it)|^4e^{-\delta t}dt\sim(2\pi^2\delta)^{-1}\log^4 1/\delta,$$

as $\delta\to 0$, which is due to Titchmarsh; and this is equivalent to

$$\int_0^{T}|\zeta(\tfrac{1}{2}+it)|^4dt\sim(2\pi^2)^{-1}T\log^4 T,$$

as $T\to\infty$, which was proved by Ingham. The proof makes use of a result of Estermann's concerning sums of the type $\sum_{m=1}^{n}d(m)d(m+k)$, and this depends on estimates for Kloosterman sums. It is pointed out by the author that the error term $O((1/\delta)^{13/14+\epsilon})$ can be replaced by $O((1/\delta)^{8/9+\epsilon})$ if better estimates for Kloosterman sums are used.

H. S. Zuckerman (Seattle, Wash.).

Zassenhaus, Hans. Tabelle der Absolutglieder der Eisensteinreihen $E_3(\tau)$ für die ersten Primzahlen und Dimensionen. Abh. Math. Sem. Hansischen Univ. **14**, 285–288 (1941). [MF 5447]

Linés Escardó, E. A theorem on the frequency of the points of a lattice which are on a strip, interior to another, both of known width. Revista Mat. Hisp.-Amer. (4) **1**, 75–81 (1941). (Spanish) [MF 5097]

The author gives a not quite adequate and rather complicated geometrical proof of a special case of the following well-known fact: Let k be irrational, $\delta>0$; then the number of solutions of $b<y-xk<b+\delta$ in integers x, y with $0\leqq x\leqq X$ is asymptotic to δX. *P. Scherk.*

Mordell, L. J. On the product of two non-homogeneous linear forms. J. London Math. Soc. **16**, 86–88 (1941). [MF 5125]

Let $a_1, b_1, c_1, a_2, b_2, c_2$ be real numbers and $a_1b_2-a_2b_1=1$. A well-known theorem of Minkowski states that the inequality

$$|(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)|\leqq\tfrac{1}{4}$$

has an integral solution x, y. A new proof is given, suggested by geometric considerations. *C. L. Siegel.*

Mordell, L. J. On the minimum of a binary cubic form. J. London Math. Soc. **16**, 83–85 (1941). [MF 5124]

Let $f(x, y)$ be an arbitrary binary cubic form with real coefficients and the discriminant d. There exist integers $x, y\neq 0, 0$ such that $|f(x, y)|\leqq k^{\frac{1}{3}}$, where $k=d/49$ for $d>0$ and $k=-d/23$ for $d<0$ (best possible constants). The proof of this theorem and of some other related results is to appear in the Proc. London Math. Soc. *C. L. Siegel.*

Davenport, H. Note on the product of three homogeneous linear forms. J. London Math. Soc. **16**, 98–101 (1941). [MF 5128]

Let L_1, L_2, L_3 be three homogeneous linear functions in x, y, z with real coefficients and determinant 1. Some time ago, the author proved [Proc. London Math. Soc. (2) **44**, 412–431 (1938)] that the lower bound of $|L_1L_2L_3|$ for integral values of $x, y, z\neq 0, 0, 0$ is not greater than $1/7$ (best possible constant). In the present note the original laborious proof is replaced by a very short and simple one, using the reduction of binary quadratic forms and a new lemma on the minimum of cubic polynomials.

C. L. Siegel (Princeton, N. J.).

Davenport, H. On a conjecture of Mordell concerning binary cubic forms. Proc. Cambridge Philos. Soc. **37**, 325–330 (1941). [MF 5541]

The author here proves the following: Let ϵ be any positive number. There exists a binary cubic form $F(x, y)$ of determinant D and with real coefficients such that none of the roots of $F(x, 1)=0$ is equivalent to any root of $\varphi(x)=0$ with the property that $F(x, y)>\alpha(\epsilon)$ for all integers $x, y\neq 0, 0$, where $\varphi(x)$ is x^3+x^2-2x-1 or x^3-x-1 according as $D<0$ or >0 and

$$\alpha(\epsilon)=\left(\frac{|D|}{49+\epsilon}\right)^{\frac{1}{3}}\quad\text{or}\quad\left(\frac{D}{23+\epsilon}\right)^{\frac{1}{3}}$$

in the respective cases. This bears on a result of Mordell's which is soon to appear in the Proc. London Math. Soc., namely: $F(x, y)\leqq\alpha(0)$ for all real binary cubics. This paper shows that Mordell's upper limit cannot be bettered even by excluding forms allied with $\varphi(x)$. *B. W. Jones.*

Pall, Gordon. The construction of positive ternary quadratic forms. Bull. Amer. Math. Soc. **47**, 641–650 (1941). [MF 5060]

A defect of earlier methods of constructing reduced positive ternary quadratic forms was that the construction would readily yield all reduced forms with determinant less than a given fixed value but was not adapted to the construction of forms with a specified determinant. The present method, utilizing in part the adjoint form, avoids this defect, and is also well adapted to the construction of a single genus. The method is quite simple and is illustrated with calculations for determinant 600 and for a genus of determinant 324. *M. Hall* (New Haven, Conn.).

Braun, Hel. Zur Theorie der hermitischen Formen. Abh. Math. Sem. Hansischen Univ. **14**, 61–150 (1941). [MF 5441]

The author obtains for Hermitian forms results analogous to those of C. L. Siegel [Ann. of Math. (2) **36**, 527–606 (1935); **37**, 230–263 (1936); **38**, 212–291 (1937)] for quadratic forms. The coefficients of the Hermitian forms are algebraic integers in an arbitrary quadratic field K over the

field of rational numbers. Siegel's methods appear to carry through to a large extent if every ideal in K is a principal ideal. For Hermitian forms over other quadratic fields it is not true, for instance, that every form is equivalent to one with a non-zero determinant. Here it is necessary to build on Siegel's results for quadratic forms with algebraic integers as coefficients and on methods of Humbert [C. R. Acad. Sci. Paris 169, 309–316, 360–365, 407–414 (1919)].

An important result of Braun's which appears to have no analogue in Siegel's work is his theorem II: Given r, β, d, an r-rowed Hermitian matrix $\mathfrak{H}$ and the modulus q with $4Md^3 \mid q$, where M is the absolute value of the fundamental number (Grundzahl) of the quadratic field K. Then there is a matrix $\mathfrak{H}_0 \sim \mathfrak{H} \pmod q$ of rank r, index β and discriminant d (d is the greatest common ideal divisor of the r-rowed minors of $\mathfrak{H}_0$) if there is an integer x in K and an integral ideal α in K such that

$$(-1)^\beta dN(x) \equiv N(\alpha)\,|\mathfrak{H}| \pmod q, \quad (x, q)=1, \ (\alpha, q)=1.$$

Furthermore one can choose $\mathfrak{H}_0$ to be an $r+1$-rowed matrix such that

$$\mathfrak{H}_0 \equiv \begin{pmatrix} \mathfrak{H} & \mathfrak{n} \\ \mathfrak{n}' & 0 \end{pmatrix} \pmod q.$$

While the basis is laid for consideration of indefinite as well as positive Hermitian forms, the theory is carried through explicitly only for the latter. Here, the result analogous to Siegel's is

$$\frac{M(\mathfrak{H}, \mathfrak{K})}{M(\mathfrak{H})} = \epsilon d^{-s} c^{r-s} \lambda \prod_{j=r-s+1}^{r} \frac{(2\pi)^j}{(j-1)!\,M^{j/2}},$$

where Hermitian forms $\mathfrak{H}$ and $\mathfrak{K}$ have ranks r and s and discriminants d and c, respectively, $r \geqq s$, $\epsilon = \frac{1}{2}$ or 1 according as $r=s$ or $r>s$, M is defined above,

$$\lambda = \lim_{q \to \infty} q^{s(s-2r)} A_q(\mathfrak{H}, \mathfrak{K})$$

and the other terms are defined as in Siegel's work.

B. W. Jones (Ithaca, N. Y.).

Koksma, J. F. und Meulenbeld, B. Diophantische Approximationen homogener Linearformen in imaginären quadratischen Zahlkörpern. Nederl. Akad. Wetensch., Proc. **44**, 426–434 (1941). [MF 5020]

In a recent paper [Nederl. Akad. Wetensch., Proc. **44**, 62–74 (1941); see these Rev. **2**, 253] the authors (transforming Blichfeldt's method) obtained a new result concerning the approximation of linear forms. Here they apply their method to obtain the following analogous result in the field $k(i\sqrt{m})$, $m \geqq 1$ (m quadrat frei). Suppose that n is a positive integer, that $\lambda=1$ for $m \not\equiv 3 \pmod 4$, $\lambda=2$ for $m \equiv 3 \pmod 4$, and let

$$\rho_{n,m} = \frac{2^{n+1}\eta^n}{(n+1)^n}\left[\left\{\frac{(2n+1)m^{(n+1)/2}}{(n+1)\pi^{n+1}\lambda^{n+1}}\right\} \Big/ \left\{1 + \frac{(n-1)^{2n+2}}{n(n+1)^{2n+1}} + \frac{2^{2n+3}\eta^{2n+1}(2n+1)}{(n+1)^{2n+1}} \cdot \sum_{\mu=2n+3}^{\infty} \frac{1}{\mu}\left(\frac{n-1}{2n}\right)^\mu\right\}\right]^{\frac{1}{2}}.$$

Then to any system of complex numbers $(\theta_1, \theta_2, \cdots, \theta_n)$ and real $t>2$, there are "integers" $(P_1, P_2, \cdots, P_n, Q)$ of $k(i\sqrt{m})$, $1 \leqq P = \max(|P_1|, |P_2|, \cdots, |P_n|) \leqq 2(\rho_{n,m} t)^{1/n}$, for which the linear form $L = \theta_1 P_1 + \theta_2 P_2 + \cdots + \theta_n P_n - Q$ satisfies $|L| \leqq \rho_{n,m}/P^n$, $|L| \leqq 2/t$.

D. C. Spencer (Cambridge, Mass.).

Koksma, J. F. und Meulenbeld, B. Simultane Approximationen in imaginären quadratischen Zahlkörpern. Nederl. Akad. Wetensch., Proc. **44**, 310–323 (1941). [MF 5014]

The authors carry over to complex numbers the fundamental method of Blichfeldt [Trans. Amer. Math. Soc. **15**, 227–235 (1914)], and obtain the following theorem on simultaneous approximation in the complex quadratic field $k(i\sqrt{m})$, where m is a positive (quadrat frei) integer. Suppose that n is a positive integer, that $\lambda=1$ for $m \not\equiv 3 \pmod 4$, $\lambda=2$ for $m \equiv 3 \pmod 4$, and let

$$\gamma_{n,m} = \frac{2n}{\pi^{\frac{1}{2}}(n+1)}\left[\left\{\frac{4(2n+1)m^{(n+1)/2}}{(n+1)\pi\lambda^{n+1}}\right\} \Big/ \left\{1 + \frac{(n-1)^{2n+2}}{n(n+1)^{2n+1}} + \frac{2^{2n+3}\eta^{2n+1}(2n+1)}{(n+1)^{2n+1}} \cdot \sum_{\mu=2n+3}^{\infty} \frac{1}{\mu}\left(\frac{n-1}{2n}\right)^\mu\right\}\right]^{1/2n}$$

Then to any system of complex numbers $(\theta_1, \theta_2, \cdots, \theta_n)$, and real $t>2$, there is at least one system of "integers" of $k(i\sqrt{m})$, $(Q, P_1, P_2, \cdots, P_n)$, say, where $1 \leqq |Q| \leqq 2t^n \gamma_{n,m}^n$, for which

$$\left|\theta_\nu - \frac{P_\nu}{Q}\right| \leqq \frac{\gamma_{n,m}}{|Q|^{1+1/n}}, \quad \left|\theta_\nu - \frac{P_\nu}{Q}\right| \leqq \frac{2}{t|Q|}, \quad \nu = 1, 2, \cdots, n.$$

D. C. Spencer (Cambridge, Mass.).

Duffin, R. J. and Schaeffer, A. C. Khintchine's problem in metric Diophantine approximation. Duke Math. J. **8**, 243–255 (1941). [MF 4589]

Let $\{\alpha_q\}$ be a sequence of positive numbers satisfying: (i) $\sum_{q=1}^{\infty} \alpha_q = \infty$; (ii) $q\alpha_q$ is a decreasing function of q. Khintchine [Math. Ann. **92**, 115–125 (1924); Math. Z. **24**, 706–714 (1926)] has proved under these circumstances that for almost all x there are infinitely many rational numbers p/q for which (1) $|x - p/q| < \alpha_q/q$. The authors suppose only that $\alpha_q \geqq 0$, $\sum \alpha_q = \infty$, replace condition (ii) by the weaker hypothesis (iii) that there is a number $c>0$ such that $\sum_{\nu=1}^{n} \alpha_\nu \phi(\nu)/\nu > c \sum_{\nu=1}^{n} \alpha_\nu$, for arbitrarily many n, where $\phi(\nu)$ is Euler's function, and prove that, for almost all x, (1) has an infinity of solutions in relatively prime p and q. An example is constructed which shows that (i) alone is not sufficient to secure the conclusions of Khintchine's theorem.

D. C. Spencer (Cambridge, Mass.).

*Hardy, G. H. **Ramanujan. Twelve lectures on subjects suggested by his life and work.** Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1940. vii+236 pp. $6.00.

From the fact that practically all topics of analytic number theory are mentioned, briefly or extensively, in this book in connection with one or the other of Ramanujan's ideas, theorems, conjectures, we realize the far-reaching influence which his work has had on present-day mathematics. Ramanujan did not invent new powerful methods nor did he introduce new notions to bring order in hitherto unrelated mathematical facts. His reputation rests much more upon the new and unexpected material with which he enriched our science, upon his famous formulae and identities derived with an incomparable skill of formal calculation or discovered by a sort of divination, the depth and precision of which still astonish us twenty years after his death.

Hardy discusses in eleven chapters some outstanding mathematical investigations of Ramanujan. The subject is

presented against the background of the general problem and its history. The evidence of Ramanujan's independence is carefully weighed, his proofs analyzed and the flaws of the reasonings which he took for proofs clearly pointed out. Usually one proof is carried through in all details and further refinements and extensions of the idea grouped around this exposition. Notes at the end of each chapter contain references, short explanations and additional information about material related to the object treated. In this way the book is not only an homage to Ramanujan's genius; it is a survey of many branches of modern arithmetic and analysis and, altogether, a book which makes fascinating reading.

The contents are as follows: I. The Indian mathematician Ramanujan. A biographical sketch; reprint of Hardy's lecture at the Harvard Tercentenary. II. Ramanujan and the theory of prime numbers. This chapter discusses Ramanujan's shortcomings in analytic function theory. The classical proof for the prime-number theorem is given, the Wiener-Ikehara proof indicated. III. Round numbers. The Hardy-Ramanujan proof ["Papers" No. 35] and Turan's proof about the normal number of factors of a number. IV. Some more problems of the analytic theory of numbers. Ingham's proof of $\zeta(1+it)\neq 0$, based on a formula of Ramanujan; Ramanujan's misleading statement about the number of numbers which are sums of two primes; an assertion of his connected with $\mu(n)$ and equivalent to the prime-number theorem. V. A lattice-point problem. The problem of lattice-points in the circle and Dirichlet's divisor problem are briefly mentioned. Details are given concerning the problem of numbers $2^a3^b\leqq n$, which is a lattice-point problem for a right-angled triangle. Pillai's theorem about the differences 2^a-3^b is proved by means of the Thue-Siegel theorem. VI. Ramanujan's work on partitions. Franklin's proof of Euler's pentagonal number theorem; divisibility of $p(5m+4)$, $p(7m+5)$, $p(11m+6)$, the first two with Ramanujan's proofs; the Ramanujan identities for

$$\sum p(5m+4)x^m \quad \text{and} \quad \sum p(7m+5)x^m.$$

The Rogers-Ramanujan identities with proofs and their arithmetical interpretation by Schur and MacMahon.

VII. Hypergeometric series. The Dougall-Ramanujan identity and consequences. Generalization to basic hypergeometric series, leading to another proof of the Rogers-Ramanujan identities. VIII. Asymptotic theory of partitions. An account of the development of the Hardy-Ramanujan theory; their determination of $\log p(n)$ by Tauberian methods sketched; MacMahon's and Lehmer's numerical tests. Detailed proof of Rademacher's series for $p(n)$. IX. The representation of numbers as sums of squares. "Ramanujan's sums," singular series. Hardy's method for determining $r_{24}(n)$, based on the theory of modular functions. New is the first detailed proof of Ramanujan's formula

$$r_{24}(n)=(16/691)\sigma_{11}^*(n)+(128/691)\{(-1)^{n-1}259\tau(n)-512\tau(\tfrac{1}{2}n)\};$$

elementary proof of $\tau(n)=O(n^3)$.

X. Ramanujan's function $\tau(n)$. Mordell's proof of the multiplicativity of $\tau(n)$ is given and leads to

$$F(s)=\sum_{n=1}^{\infty}\tau(n)/n^s=\prod_p(1-\tau(p)p^{-s}+p^{11-2s})^{-1}.$$

I regret that Hecke's recent work on modular forms and prime numbers [Danske Vid. Selsk. Math.-Fys. Medd. 13, no. 10 (1935); Math. Ann. 114, 1–28, 316–351 (1937)] is not mentioned either here or in connection with the functional equation

$$(2\pi)^{-s}\Gamma(s)F(s)=(2\pi)^{s-12}\Gamma(12-s)F(12-s).$$

Ramanujan's formulae would have appeared as a special instance of Hecke's theory. Hardy then discusses congruence properties of $\tau(n)$. In a paper to be published in Amer. J. Math., D. H. Lehmer remarks that $\tau(23)\not\equiv 0 \pmod{23}$, contrary to Hardy's statement. The proof of $\tau(23m+k)\equiv 0 \pmod{23}$ for all k which are quadratic non-residues mod 23 is not affected by this correction. Then $\tau(n)\equiv 0 \pmod 5$ for almost all n is proved and a proof indicated for the modulus 691. The rest of the chapter is devoted to the order of $\tau(n)$, in particular to Rankin's proof of $\tau(n)=O(n^{6-\frac{1}{2}})$. Ramanujan's conjecture of $O(n^{11/2+\epsilon})$ is still unproved.

XI. Definite integrals. Theorems which are "inevitably less impressive" than much of Ramanujan's other work. He could not have possessed any "real proof" of these theorems since he neglected the natural conditions of their validity. The integrals under discussion are connected with Mellin's formula, with Fourier transforms and with Poisson's and Plana's remainder of the Euler-Maclaurin sum formula. XII. Elliptic and modular functions. Ramanujan's investigations here are based on the infinite product

$$\prod(a,b)=(1+a)(1+ab)(1+ab^2)\cdots.$$

"In this notation many fundamental formulae appear in odd disguises." The modular equation of order 3 for the Legendre modulus k is derived, first by means of Jacobi's transformation, then through ϑ-functions, then in Ramanujan's manner. An exposition of the theory of singular moduli follows [no. 12.18 contains a mistake about the degree of the modular equation $F(x, J)=0$ of order m]. Some of Watson's reconstructions of Ramanujan's computations are sketched.

The book closes with a bibliography of books and papers dealing with Ramanujan and his work. *H. Rademacher.*

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Foster, Alfred L. Natural systems: the structure of abstract monotone sequences. Proc. Nat. Acad. Sci. U. S. A. 27, 325–327 (1941). [MF 4584]

By a natural system, with unit, the author understands an abstract system $(N, \circ)$, where N is a denumerable set and $\circ$ a binary operation (product), such that the product of two elements of N is unique and in N, the associative and commutative laws hold, there is a single unit, there is at least one prime, and unique factorization into primes holds. A regular ordering of N is an arrangement of all the elements of N as an infinite sequence $e_1, e_2, \cdots$ such that, if $m<n$, then $ee_m<ee_n$ for every element e of N. The paper announces, without proof, a complete characterization of a regular ordering, with the aid of a depiction of N in a real space. *H. Blumberg* (Columbus, Ohio).

Dushnik, Ben and Miller, E. W. Partially ordered sets. Amer. J. Math. 63, 600–610 (1941). [MF 4684]

The authors are interested in the representations of a given partial ordering P of a fixed set as intersections $\prod L_i$ of linear orderings L_i of the set. Every partial ordering has such a representation; the smallest number of L_i such that $P=\prod L_i$ may be called the dimension of P and denoted $d(P)$. The authors prove that $d(P)\leqq 2$ if and only if P can be realized by intervals of a linearly ordered set, partially ordered by set-inclusion. The existence of partial orderings of every dimension is proved. Other results are given.

G. Birkhoff (Cambridge, Mass.).

Denjoy, Arnaud. Représentation conjointe de l'ordination et de l'énumération d'un ensemble dénombrable, par un nombre ou par une figure plane. C. R. Acad. Sci. Paris 212, 885–888 (1941). [MF 5036]

The author studies the family of all enumerations and orderings of an enumerable set E. To each enumeration and ordering of E a number ξ on $[0, 1]$ is assigned in such a way that to each irrational ξ there is just one enumeration and ordering of E and to each rational ξ there are two. Given an ordered $E=(\theta_n)$, let $e_{n-1}=(\theta_0, \theta_1, \cdots, \theta_{n-1})$ and define a_n as the number of $\theta_i \varepsilon e_{n-1}$ such that $\theta_i<\theta_n$. The number $\xi=\sum a_n/(n+1)!$ is assigned to E. Conversely, to each such ξ with a_n integral, $0\leqq a_n\leqq n$, there is an ordering and enumeration of E to which ξ is assigned. The a_n are used to define a family of plane polygonal lines which give a geometric characterization of the enumeration and ordering of E.

L. W. Cohen (Lexington, Ky.).

Sierpiński, W. Sur les espaces métriques universels. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 75, 571–574 (1940). [MF 4787]

A proof is indicated for the theorem: if τ is an ordinal greater than 0 such that $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ for $0\leqq\alpha<\tau$, there exists a metric space of power $\aleph_\tau$ containing an isometric image of any metric space of power $\aleph_\tau$.

L. W. Cohen.

Sierpiński, W. Sur un espace métrique séparable universel. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 75, 575–577 (1940). [MF 4788]

The author indicates a simple proof of the theorem of Banach and Mazur that a separable metric space M is isometric with a subset of the space C of continuous functions on $[0, 1]$. Let (p_n) be dense in M and let $x_n=\phi_n(t)$ be a continuous mapping of $[0, 1]$ into the cube $0\leqq x_n\leqq 1$, $n=1, 2, \cdots$. Let $t_k \varepsilon [0, 1]$ be a solution of $\rho(p_k, p_1)-\rho(p_k, p_n)=\rho(p_1, p_n)[2\phi_n(t)-1]$ for all n, where ρ is the metric on M. Let $f_n(t)=\rho(p_1, p_n)[2\phi_n(t)-1]$ on the closure of $T=(t_k)$; f_n is extended linearly in the intervals contiguous to $\bar{T}$ and parallel to the t-axis on the intervals of $[0, 1]$ beyond the extremes of $\bar{T}$. For each $p \varepsilon M$ and (p_{n_k}) converging to p, $f_{n_k}(t)\to f^{(p)}(t)$ uniformly in $[0, 1]$. The mapping $p \leftrightarrow f^{(p)}(t)$ is isometric on M to C. The details of this and the preceding note should have appeared in Fund. Math. 33.

L. W. Cohen (Lexington, Ky.).

Guareschi, Giacinto. Alcune osservazioni sul comportamento di un insieme puntuale intorno ai suoi punti di accumulazione. Boll. Un. Mat. Ital. (2) 2, 414–420 (1940). [MF 5558]

A result proved by F. Severi [Ann. Mat. Pura Appl. (4) 13, 21–23 (1934)] for Jordan surfaces with only "ordinary" points is generalized by the author to certain perfect point sets of a Euclidean space R_n. The author introduces the following definitions: A limit point p of a point set P in R_n has the "character k ($<n$) of projectability" if the part of P which is contained in a sufficiently small neighbourhood of p can be projected (1, 1) in a R_k, but not in a R_i with $i<k$. If p has not any character $k<n$, then p is given the character n. According to F. Severi, the "tangent space" of P in p is the linear space of minimal dimension containing all straight lines through p which in every neighbourhood of p are limits of secants of P. If the character of p is equal to the dimension of the tangent space of P in p, then p is called a "homocharacteristic point" ("punto omocaratterico"). If P is perfect and all points of P are homocharacteristic with the same character k, then P is said to have "homogeneous homocharactersitic structure" (h.h.str.). The author then proves: A perfect set P of h.h.str. has a continuously variable tangent space.

A. Rosenthal (Ann Arbor, Mich.).

Piccard, Sophie. Sur les ensembles de distances. C. R. Acad. Sci. Paris 210, 780–783 (1940). [MF 4890]

A statement, without proofs, of the principal results of the author's monograph "Sur les ensembles de distances," Neuchâtel, 1939 [cf. these Rev. 2, 129]. *H. Blumberg.*

Schiffer, Menahem. On the subadditivity of the transfinite diameter. Proc. Cambridge Philos. Soc. 37, 373–383 (1941). [MF 5545]

Let M_1 and M_2 be two Jordan curves without common points and not containing each other. Let d_1, d_2, d be the transfinite diameters of the sets M_1, M_2, M_1+M_2, respectively. Then $d\leqq d_1+d_2$. This interesting inequality is proved first for the case M_1 and M_2 are analytic curves; the general case follows by a limiting process using Carathéodory's theorem.

G. Szegö (Stanford University, Calif.).

Zahorski, Z. Punktmengen, in welchen eine stetige Funktion nicht differenzierbar ist. Rec. Math. [Mat. Sbornik] N. S. 9 (51), 487–510 (1941). (Russian. German summary) [MF 5498]

The following results are given: If N is the set of points in which a function $f(x)$ has no derivative, then $N=N_1+N_2+N_3$, where N_1 is a G_δ, N_2 a $G_{\delta\sigma}$ of measure zero and N_3 of measure zero and such that $N_3 \subset K+\sum K_n$, where K is the set of discontinuities of $f(x)$ and $K_n=E_x[x\pm 1/n \subset K]$. Hence the above decomposition is a necessary condition in order that N be a set where a general $f(x)$ can have no derivative. For a $f(x)$ with an enumerable set of discontinuities it is shown that the above decomposition with $N_3=0$ is necessary. For a continous $f(x)$, $N_3=0$ is also sufficient (an $f(x)$ is constructed). For a function of bounded variation $N_1=N_3=0$ is necessary and sufficient. The above results are true whether we consider the points with $f'(x)=\infty$ or $-\infty$ to be points of non-differentiability or not. Sets related to derivatives of functions are investigated and numerous examples of functions having different types of sets, such as sets of non-differentiability, are constructed.

F. Wolf (St. Paul, Minn.).

Offord, A. C. Note on continuous independent functions. Quart. J. Math., Oxford Ser. 12, 86–88 (1941). [MF 5220]

Two functions $f(x)$ and $g(x)$ defined in the interval $0\leqq x\leqq 1$ are called independent if for any numbers α, β, α', β' the sets $E_1=E_x\{\alpha\leqq f(x)\leqq\beta\}$ and $E_2=E_x\{\alpha'\leqq f(x)\leqq\beta'\}$ have the following property: the measure of the product

E_1E_2 is equal to the product of the measures of E_1 and E_2 [cf. M. Kac, Studia Math. 6, 46–58 (1936); A. Kolmogoroff, C. R. Acad. Communiste 1929, 8–21]. It is known that there exist continuous functions which are independent [Steinhaus, Comment. Math. Helv. 9, 166–169 (1936)]. Here it is shown that, if two functions $f(x)$ and $g(x)$ defined in (0, 1) are continuous and neither is constant, and if $f(x)$ takes on one of its values at most a denumerable number of times, then the functions $f(x)$ and $g(x)$ cannot be independent. *A. Zygmund* (South Hadley, Mass.).

Motchane, Léon. Sur la continuité des fonctions à variation bornée continue par rapport à chacune des variables. C. R. Acad. Sci. Paris 211, 61–62 (1940). [MF 5336]

Let $f(x, y)$ be continuous for $a \leqq x \leqq b$; $c \leqq y \leqq d$. Suppose that (1) $f(x, y)$ is continuous in each variable separately; (2) for each y it is of bounded variation in x; (3) for each x, $T_a^x f(s, y)$ is continuous as a function of y. Then it is proved that $f(x, y)$ is continuous in both variables simultaneously. It is easily seen that it suffices to require (3) for $x=b$ alone. *J. A. Clarkson* (Philadelphia, Pa.).

Roger, Frédéric. Sur l'indétermination de certaines limites. C. R. Acad. Sci. Paris 212, 66–68 (1941). [MF 4892]

Let V be a p-dimensional variety in E_n, and M a point of V. Call σ the set of all $(n-p)$-spheres which are limits of $(n-p)$-spheres through M and $r=n-p+1$ points of V which tend to M. Then at all points M, except for the points of a subset of V with p-dimensional measure 0, the set σ consists of all $(n-p)$-spheres passing through a ν-sphere, $\nu=0, \cdots, n-p$, where ν depends on M. The theorem stated by the author (with hardly any indication for a proof) is even more general than this, in as much as the spheres may be replaced by a family of $(n-p)$-dimensional analytic varieties, for which a fixed r exists such that exactly one variety of the family passes through $r+1$ given points. *H. Busemann* (Chicago, Ill.).

Vicente Gonçalves, J. Sur quelques théorèmes classiques. Revista Fac. Ci. Univ. Coimbra 9, 19 pp. (1941). [MF 5556]

This paper discusses the class of real-valued functions $g(x)$ on $a \leqq x \leqq b$ which satisfy the following conditions: (1) they are finite derivatives of functions $G(x)$; (2) they have a finite or infinite derivative at every interior point of (ab); (3) their points of discontinuity constitute a reducible set. A number of properties of functions of this class are established; among them we mention the following: functions $g'(x)$ take on (ab) every value λ between their least upper and greatest lower bounds (extension of the theorem of Darboux); they do not have discontinuities of the first kind; a point, at which $g'(x)=\lambda$, is interior to arcs whose chords have slope λ; if the tangent at a point P is not parallel to the chord of any arc which contains P, then P is a point of inflexion. The author also considers "fonctions remplissantes" on the interval (ab), that is, arbitrary-valued functions which take, on every subinterval $(\alpha\beta)$ of (ab), all values between $f(\alpha)$ and $f(\beta)$. [These are the functions called "Darboux'sche Funktionen" by Hahn; the term "gapless functions" is suggested by Schoenberg.] Moreover, a function is called gapless at a point c, if, in every interval $(c, c+\epsilon)$, $\epsilon>0$ or <0, it takes every value between $\lim_{\epsilon\to 0} l(c, \epsilon)$ and $\lim_{\epsilon\to 0} L(c, \epsilon)$, where $l(c, \epsilon)$ and $L(c, \epsilon)$ are, respectively, the greatest lower and least upper bounds of the function on $(c, c+\epsilon)$. It is shown that a function is gapless on (ab) if it is gapless at every point of this interval, and conversely. *A. Dresden* (Swarthmore, Pa.).

Liberman, Joseph. Théorème de Denjoy sur la dérivée d'une fonction arbitraire par rapport à une fonction continue. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 221–236 (1941). (French. Russian summary) [MF 4496]

If $f(x)$ and $\phi(x)$ are finite real functions defined in (0, 1), the Dini derivatives of f with respect to ϕ are

$$\bar{D}_\phi{}^+ f = \overline{\lim_{x'\to x}} \frac{f(x')-f(x)}{\phi(x')-\phi(x)}, \qquad \phi(x')>\phi(x);$$

and $\underline{D}_\phi{}^+ f$, $\bar{D}_\phi{}^- f$, $\underline{D}_\phi{}^- f$ defined correspondingly. It is assumed that the set E of points of discontinuity of ϕ is of measure 0. The author proves the following theorem, analogous to that of Denjoy: For almost every x of (0, 1) the four derivatives exist and either (a) the upper derivatives equal $+\infty$, the lower $-\infty$; or (b) a lower derivative equals $-\infty$, the upper on the other side $+\infty$, and the other two derivatives are finite and equal; or (c) the four derivatives are finite and all equal. Conversely, a necessary and sufficient condition that this theorem be valid for the pair (f, ϕ) for arbitrary f is that E be of measure 0. *H. Blumberg.*

Kametani, Shunji. Theorems on interval-functions and h-measures. Jap. J. Math. 17, 533–539 (1941). [MF 5646]

The author generalizes some results of Besicovitch [Proc. London Math. Soc. (2) 32, 1–9 (1931)] and of Saks and Zygmund [Ann. Scuola Norm. Super. Pisa 3, 157–179 (1934)]. The main generalization consists in replacing α-dimensional measures by general Hausdorff measures. As an application the following extension of known results [see the papers cited] is obtained. Suppose that a continuous function $f(z)$ defined in a plane domain G satisfies the conditions: (i) for some non-decreasing $h(t)$, with $t^2/h(t)\to 0$ as $t\to+0$, and for every $z\in G$,

$$|f(z+\Delta z)-f(z)| = O\{h(|\Delta z|)\}, \qquad \Delta z\to 0;$$

(ii) $\limsup_{\zeta\to z} |(f(z)-f(\zeta))/(z-\zeta)| < \infty$ everywhere in G, except perhaps in a set expressible as a sum of a denumerable sequence of sets of finite h-measure; (iii) $f'(z)$ exists and is finite almost everywhere in G. Then $f(z)$ is regular in G. *A. Zygmund* (South Hadley, Mass.).

Reichelderfer, P. and Ringenberg, L. The extension of rectangle functions. Duke Math. J. 8, 231–242 (1941). [MF 4588]

The authors show that a simple semi-additive property is necessary and sufficient that the function have a completely additive extension on a closed range (that is, a completely additive and complementative class of sets including open sets). In the Euclidean plane a rectangle is said to be oriented if each side is parallel to an axis. Let R be an open oriented rectangle, let C be the class of oriented open rectangles in R, let $\mathfrak{C}$ be the condition: If $r_1, r_2, \cdots$ is a finite or denumerable sequence of mutually exclusive rectangles each in C, and $R_1, R_2, \cdots$ is a finite or denumerable sequence of rectangles each in C such that $\sum r_n \subset \sum R_n$, then $\sum \varphi(r_n) \leqq \sum \varphi(R_n)$. One of the theorems proved is: A necessary and sufficient condition that a function φ defined on C to the real number system have a completely additive extension to a closed range in R is that φ satisfy $\mathfrak{C}$. Other theorems are obtained for rectangles that are merely open, oriented and closed, or merely closed. *J. F. Randolph.*

Best, E. On sets of fractional dimensions. II. Proc. Cambridge Philos. Soc. **37**, 127–133 (1941). [MF 4239]

An inequality is established relating certain constants of the series $\sum_2^\infty \xi_r^{-1}$ to the dimensional number (in the sense of Besicovitch) of the set E of numbers of the form

$$x=x_2/2!+x_3/3!+x_4/4!+\cdots, \quad 0\leqq x_r<[\xi_r]+1.$$

[The first part appeared in the same Proc. **36**, 152–159 (1940); cf. these Rev. **1**, 302.] *W. Hurewicz.*

Good, I. J. The fractional dimensional theory of continued fractions. Proc. Cambridge Philos. Soc. **37**, 199–228 (1941). [MF 4956]

The author considers the fractional dimension of sets defined by continued fractions. Among others he proves the following results: (1) The set of continued fractions whose partial quotients tend to infinity has dimensional number $\frac{1}{2}$. (2) The set of continued fractions for which the nth root of the nth partial fraction is unbounded has dimensional number $\frac{1}{2}$. The only previous literature on the subject seems to be a paper by Jarnik [Prace Mat.-Fiz. **36**, 91–106 (1928)]. Jarnik is concerned with the set E of continued fractions whose partial quotients are bounded, and with the sets E_α whose partial quotients are not greater than α. He proves that $\dim E=1$. He also shows that $\dim E_2>\frac{1}{4}$. The author improves this estimate very considerably; he shows that $.5306<\dim E_2<.5320$. *P. Erdös* (Philadelphia, Pa.).

Linés Escardó, E. On the mean measure of an unbounded linear set. Revista Mat. Hisp.-Amer. (4) **1**, 43–51 (1941). (Spanish) [MF 4726]

Let C be an unbounded set of points on the interval $0\leqq x<+\infty$, Lebesgue measurable in every finite subinterval, and $\phi(x)$ the characteristic function of C. The mean measure of C is

$$\sigma=\lim_{t\to+\infty}\left(\int_0^t \phi(x)\alpha(x)dx \Big/ \int_0^t \alpha(x)dx\right),$$

where

$$\int_0^{+\infty}\alpha(x)dx=+\infty, \quad \int_0^t |\alpha(x)|dx<K\left|\int_0^t \alpha(x)dx\right|,$$

K a finite constant (independent of t). Let E denote the set of all the points x of $(0, 1)$ such that C contains infinitely many points which are congruent to x (mod 1). The first theorem states that the measure of E is equal to σ. Let C_i be the subset of C in $(i, i+1)$ $(i=1, 2, 3, \cdots)$. Let $0\leqq x\leqq 1$. Then the frequency of x is defined by

$$f(x)=\lim_{n=\infty}\left(\sum_{i=1}^n \alpha_i\epsilon_i \Big/ \sum_{i=1}^n \alpha_i\right), \quad \alpha_i>0;\ \sum\alpha_i=+\infty,$$

where $\epsilon_i=1$ or 0 according as x is or is not congruent (mod 1) to a point in C_i. Theorems are given relating frequency and mean measure. The formulations of the author are not sufficiently precise. *H. S. Wall* (Evanston, Ill.).

*Douglas, Jesse. **Survey of the theory of integration.** Galois Lectures, Scripta Mathematica Library, no. 5, pp. 1–47. New York, 1941.

A non-technical survey of the theory of integration. The starting point is the quadrature of a circle and of a segment of a parabola by Archimedes. The definitions of the Riemann, Riemann-Stieltjes and Lebesgue integrals are given, and their properties described. There is a brief mention of Fourier series and transforms, Denjoy integrals and the double integrals of Riemann and of Lebesgue.

R. L. Jeffery (Wolfville, N. S.).

Sastry, B. S. The limiting points of the set of the upper and the lower sums in Riemann integration. Math. Student **8**, 148–150 (1940). [MF 5000]

The author establishes the following theorem: Let M and m be the upper and the lower bounds, U and L the upper and the lower Riemann integrals, respectively, of $f(x)$ on $a\leqq x\leqq b$. Then every point between U and $M(b-a)$ is a limiting point of the set E of all upper Riemann sums of $f(x)$ on $a\leqq x\leqq b$; similarly, every point between L and $m(b-a)$ is a limiting point of the set e of lower sums. The proof shows how a subdivision is to be chosen so that the corresponding upper sum is arbitrarily close to a prescribed limiting point of E; similarly for lower sums. *G. B. Price.*

Reid, William T. Green's lemma and related results. Amer. J. Math. **63**, 563–574 (1941). [MF 4679]

The main result of the paper is: Suppose that R is the interior of a simply closed rectifiable plane curve J, and that $M(x, y)$ is such that (i) it is continuous on $R+J$; (ii) if $[a, b; c, d]$ is any rectangle in R, $M(x, y_0)$ is absolutely continuous in x on $[a, b]$ for y_0 almost everywhere on $c\leqq y\leqq d$; (iii) $\partial M/\partial x$ is summable on R. Then

$$\int\int_R (\partial M/\partial x)dxdy=\int_J M(x, y)dy.$$

The writer has succeeded in reducing the proof to about as elementary considerations as possible, thus making a satisfactory and rigorous proof now accessible, with a minimum of reference to any but the more familiar parts of the theory of point sets or of analysis. The proof rests upon a limiting process applied to the Green's lemma for a region bounded by a simple polygon J', in $R+J$, having essentially the following properties: (i) each straight line segment of J' is parallel to either the x- or y-axis; (ii) the length of J' does not exceed five times the length of J; (iii) each point of J' lies in the ϵ-neighborhood of J, ϵ being an arbitrary positive number. The existence of such a polygon is proved as a fundamental lemma.

The author applies the theorem to the proof of Cauchy's integral theorem for a function $f(z)$ holomorphic in R and continuous on J, also to the proof, under somewhat more general conditions, of a result announced by Hestenes [Bull. Amer. Math. Soc. **45**, 519 (1939)]. *H. E. Bray.*

Tchélidzé, W. Un théorème sur l'intégrale de surface. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **7**, 105–112 (1940). (Russian. French summary) [MF 5305]

Let A, B, C be functions continuous on a region G of (x, y, z)-space. Assume that for all functions $z(x, y)$ which are of class C^1 on a region R and assume given boundary values the integral $\int\int_R [A+Bz_x+Cz_y]dxdy$ has the same value. If $\partial A/\partial z$ exists and is majorized by a summable function of x and y, then for every square Q in R and every constant z_0 such that the set $[(x, y)\varepsilon Q,\ z=z_0]$ is in G the integral of $A_z(x, y, z_0)$ over Q is equal to the line integral of $Cdx-Bdy$ around the boundary of Q. In particular, if the given integral is the first variation of a double integral

$\int\int F(x, y, z, p, q)dxdy$, this theorem leads to Haar's equations. The proof is made by constructing for small positive h the function which in Q is the smaller of the quantities z_0+h and z_0+distance to boundary of Q. This is independent of h by hypothesis, and for $h=0$ its derivative with respect to h is the difference between the integral of A_z over Q and the line integral mentioned. As a corollary, if for each fixed z the partial derivative $B_z(x, y, z)$ exists and is majorized by a summable function of y alone, then C_y exists almost everywhere in G and $A_z=B_x+C_y$. The theorems actually need some unspecified assumptions concerning the structure of G. Some simple ones suggest themselves, adequate for the derivation of Haar's equations.

E. J. McShane (Charlottesville, Va.).

Theory of Functions of Complex Variables

Belinfante, M. J. Elemente der intuitionistischen Funktionentheorie. II, III. Der Satz vom Integral der logarithmischen Ableitung. I, II. IV. Der Weierstrasssche Unbestimmtheitssatz. V. Die intuitionistische Uebertragung des Picardschen Satzes. Nederl. Akad. Wetensch., Proc. **44**, 276–285, 420–425, 563–567, 711–717 (1941). [MF 5010, 5019, 4975, 5117]

[The first communication appeared in the same Proc. **44**, 173–185 (1941); cf. these Rev. **2**, 355.] The second and third communications lead up to the intuitionistically correct form of the theorem expressing the number of zeros of a regular function $f(z)$ in a simply connected region G by the boundary integral J_G of $(2\pi i)^{-1}f'(z)/f(z)$. The boundary of G is supposed to be a "path" $F(t)$, where $F(t)$ is a regular function in the sense of the first communication defined in an interval $a\leqq t\leqq b$. The function $f(z)$ is assumed to be regular in a circle $|z|\leqq R$ containing G and $f(z)\neq 0$ on the boundary of G. Then J_G is an integer; moreover $J_G\geqq 0$ provided $f(z)$ is "variable," that is, provided I know two points z_1 and z_2 in the circle such that $f(z_1)$ and $f(z_2)$ are unmistakably distinct. If $J_G=0$ and $|f(z)|\geqq k>0$ on the boundary of G then $|f(z)|\geqq k$ everywhere in G. Finally $J_G=m>0$ implies that $f(z)$ has m well-located zeros in G, that is, there exist m points $z_1, \cdots, z_m$ in G such that $f(z)/(z-z_1)\cdots(z-z_m)$ is regular and not equal to 0 in G. While the reasoning in the first communication followed closely the well-known classical procedures after all desirable conditions of uniformity had been included in the definitions, these further results require more subtle and novel arguments. An essential tool is the decomposition of G into arbitrarily small sub-regions on whose boundaries $f(z)$ does not vanish.

The fundamental theorem of algebra, including Brouwer's supplement, that $a_0z^n+\cdots+a_n$ has at least $n-k$ zeros when it is known that $a_k\neq 0$, is obtained as an immediate consequence. This and the fact that a function in an arbitrarily small neighborhood of an essential singularity comes arbitrarily close to any preassigned value are proved in article IV, while V deals with what is the principal goal of this whole investigation: Picard's theorem (in conjunction with Bloch's and Schottky's theorems). The intuitionistic form of Picard's "great" theorem is as follows: Let $f(z)$ be a function with an isolated essential singularity at the origin, and a, b two distinct numbers; in any circle around the origin $f(z)$ assumes at least one of the two values a, b.

H. Weyl (Princeton, N. J.).

Doob, J. L. A minimum problem in the theory of analytic functions. Duke Math. J. **8**, 413–424 (1941). [MF 5179]

The main result of this paper is: Let $f(z)$ be a Lebesgue integrable function, defined on γ: $|z|=1$. There is one and only one function $p(z)$ of power series type that minimizes the integral $\int_\gamma |f(z)-p(z)||dz|$. The nature of the minimizing function is also determined. For the special case when $f(z)$ is a polynomial in $1/z$ the problem has been solved by F. Riesz in Acta Mathematica **42** (1920).

O. Szász (Cincinnati, Ohio).

Vicente Gonçalves, J. Contours de Jordan et intégrale de Cauchy. Portugaliae Math. **2**, 166–172 (1941). [MF 5224]

A proof of the Jordan curve theorem based on some refinements of the familiar method of polygonal approximation and the inclusion of the curve in a polygonal ring. The Schoenfliess accessibility theorem is obtained by setting up a parametric representation of the polygonal approximations which tends to the parametric representation of the curve. An integral of a continuous function along the curve can then be looked upon as the limit of this integral taken along the boundary of the polygonal regions. Thus a proof is obtained of the Cauchy-Goursat theorem which does not require the analyticity of the curve. The details of the method do not lend themselves to a brief summary.

A. Dresden (Swarthmore, Pa.).

Fabry, Eugène. Sur les séries les plus générales. C. R. Acad. Sci. Paris **211**, 245–247 (1940). [MF 5348]

The author attempts to formulate new reasons, of a somewhat intuitive character, for the well-known statement, pointed out by him almost fifty years ago [Ann. École Norm. **13**, 367 (1896)], that "in the general case" the circle of convergence of a Taylor series with a finite radius of convergence is a cut. Corresponding precise statements, according to various points of view, have been given in the mean time by many authors.

S. Mandelbrojt.

Szegö, G. Power series with multiply monotonic sequences of coefficients. Duke Math. J. **8**, 559–564 (1941). [MF 5192]

A sequence $\{a_n\}$ is called monotonic of order k if

$$\Delta^{(\nu)}a_n=a_n-\binom{\nu}{1}a_{n+1}+\binom{\nu}{2}a_{n+2}-\cdots+(-1)^\nu\binom{\nu}{\nu}a_{n+\nu}\geqq 0,$$
$$\nu=0, 1, 2, \cdots, k;\ n=0, 1, 2, \cdots.$$

L. Fejér has shown [Trans. Amer. Math. Soc. **39**, 18–59 (1936)] that the power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is regular and univalent for $|z|<1$ if the sequence $\{a_n\}$ be monotonic of order 4. The author, in a letter to L. Fejér, gave an example showing that the theorem is not true for sequences monotonic of order 2. At a later time S. Sidon [Acta. Litt. Sci. Szeged **9**, 244–246 (1940); these Rev. **1**, 213] gave another counter-example with different proof.

In this paper the author shows that Fejér's theorem above is also true for monotonic sequences of order 3, and publishes his counter-example to show that it is not true for monotonic sequences of order 2. His proof depends upon a detailed analysis of the univalent character in the unit circle of the polynomials

$$\binom{n+2}{2}+\binom{n+1}{2}z+\binom{n}{2}z^2+\cdots+z^n.$$

This univalency is of a type that yields univalency for linear sums of these polynomials with non-negative coefficients.

M. S. Robertson (New Brunswick, N. J.).

Rios, Sixto. On disarranged power series and the overconvergence of a class of Dirichlet series. Revista Mat. Hisp.-Amer. (3) **2**, 1–7 (1940). (Spanish) [MF 5045]

The author deals with series of the type $\sum a_n z^{\lambda_n}$, where $\{\lambda_n\}$ is a sequence of positive integers which do not increase to infinity. Here are some properties of such series: the radius of holomorphy and that of overconvergence are equal, but the radius of convergence is not necessarily equal to the radius of overconvergence. A theorem analogous to Ostrowski's theorem characterizing the overconvergent Taylor series (in a neighborhood of a regular point on the circle of convergence) is given for a certain type of disarranged power series. The author studies also the problem of overconvergence for some Dirichlet series, where the sequence $\{\lambda_n\}$ is of infinite maximum density.

S. Mandelbrojt (Houston, Tex.).

Boas, R. P., Jr. A note on functions of exponential type. Bull. Amer. Math. Soc. **47**, 750–754 (1941). [MF 5480]

A short and simple proof is given for a recent theorem of Widder [Proc. Nat. Acad. Sci. U. S. A. **26**, 657–659 (1940); cf. these Rev. **2**, 219. See also these Rev. **2**, 351.]. Incidentally new proofs for results by Poritsky, Whittaker and Schoenberg concerning functions representable by a Lidstone series are also obtained. *I. J. Schoenberg.*

Wright, E. M. The asymptotic expansion of integral functions defined by Taylor series. II. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 217–232 (1941). [MF 5549]

[The first part appeared in the same Trans. **238**, 423–451 (1940); these Rev. **1**, 212.]. The author extends to "best possible" the results he has previously obtained. He obtains asymptotic expansions for the function $f(x)$, with Taylor series

$$f(x)=\sum_{n=0}^{\infty}\frac{\phi(n)x^n}{\Gamma(\kappa n+\beta)}, \qquad \Re(\kappa)>0,$$

for large x when $\phi(t)$ satisfies certain conditions. In the first place $\phi(t)$ must be regular in the sector $-\mu_1\leqq\arg \kappa t\leqq\mu_2$, $|t|>K$, where μ_1, μ_2 satisfy $-\frac{1}{2}\pi<-\mu_1\leqq\gamma\leqq\mu_2<\frac{1}{2}\pi$, γ being $\arg\kappa$. In addition there must exist an $M>0$ and numbers $\alpha_1,\cdots,\alpha_{M+1}, A_1,\cdots,A_M$, such that $\Re(\alpha_1)\leqq\Re(\alpha_2)\leqq\cdots\leqq\Re(\alpha_M)<\Re(\alpha_{M+1})$ and such that, in the sector described above,

$$\frac{\phi(t)}{\Gamma(\kappa t+\beta)}=\sum_{m=0}^{M}\frac{\kappa A_m}{\Gamma(\kappa t+\alpha_m)}+O\left(\frac{1}{\Gamma(\kappa t+\alpha_{M+1})}\right).$$

Under these conditions on $\phi(t)$ the author obtains a region in the x-plane where an asymptotic expansion for $f(x)$ can be obtained. For a suitable choice of $x^{1/\kappa}$ this region is given by $-\min(\mu_1,\omega_2)\leqq\arg x^{1/\kappa}\leqq\min(\mu_2,\omega_1)$, where $\omega_1=\omega(\mu_1,-\gamma)$ and $\omega_2=\omega(\mu_2,\gamma)$ and $\omega(\mu,\gamma)$ is defined uniquely, as the author shows, by

$$\cos\omega=\cos\mu e^{(\omega+\mu)\tan\mu-2\pi\delta^{-1}\sec\mu\sin(\omega-\gamma)}, \qquad -\mu\leqq\omega<\tfrac{1}{2}\pi,$$

where $|\delta|=|\kappa|$. The $\arg x$ is chosen to satisfy $-\pi<\arg x-\tan\gamma\log|x|\leqq\pi$. Among other things the author shows that, if, for some integer s, $X_s=x^{1/\kappa}e^{2\pi i s/\kappa}$ satisfies $-\min(\mu_1,\omega_2)\leqq\arg X_s\leqq\min(\mu_2,\omega_1)$, then

$$f(x)=X_s e^{X_s}\left(\sum_{m=1}^{M}A_m X_s^{-\alpha_m}+O(X_s^{-\alpha_{M+1}})\right).$$

N. Levinson (Cambridge, Mass.).

Hibbert, Lucien. Réseau $\log R=$const., $V=$const., des fonctions entières. J. Math. Pures Appl. (9) **19**, 395–409 (1940). [MF 4630]

In the present paper the author continues and systematizes the work done in a series of articles [C. R. Acad. Sci. Paris **205**, 1121–1123 (1937); **209**, 287–289, 718–720, 783–786 (1939); **210**, 35–37 (1940); cf. these Rev. **1**, 56, 122, 211, 307] and in his thesis [Paris, 1938] on the nets of curves $\log R=$const. and $V=$const., where $Re^{iV}=f$ is an entire function whose essential singularity has been removed to a finite position A. Since the curves may have multiple points at zeros of f and of f', rules are set up for following a given branch through such a point. Under these rules, curves $\log R=$const. either close or abut at A when traversed in either direction, and curves $V=$const. abut at A or at a zero of f. The author studies the topological properties of these curves with respect to the zeros of f and f' and to A, and examines the behavior of V on the curves $\log R=$const. and that of R on the curves $V=$const. in various cases. The ultimate purpose of the investigation is to throw light on the behavior of the entire function f at its essential singularity; however, this and other properties of the curves are left by the author for a subsequent memoir. *J. W. Green.*

Dassen, Claro Cornelio. Concerning an academic communication. An. Soc. Ci. Argentina **132**, 3–21 (1941). (Spanish) [MF 5613]

Rejoinder to reviews [Revista Union Mat. Argentina **7**, 61–64 (1941)] of papers of Blaquier, Dassen and Biggeri [see these Rev. **2**, 183]. *R. P. Boas, Jr.* (Durham, N. C.).

Biggeri, Carlos. A new general proof of the second theorem of Picard. An. Soc. Ci. Argentina **132**, 78–79 (1941). (Spanish) [MF 5614]

The proof uses Bloch's theorem. Its brevity is made possible by use of the statement in lines 8 to 12 of the paper; this statement seems to the reviewer to require proof.

R. P. Boas, Jr. (Durham, N. C.).

Ganapathy Iyer, V. On periodic integral functions. J. Indian Math. Soc. (N.S.) **5**, 1–17 (1941). [MF 5108]

The author demonstrates several theorems on periodic integral functions some of which are a consequence of the following simple theorem on the representation of periodic functions which he proves: Let $f(z)$ be a periodic integral function with period $\lambda>0$. Let $\{z_n^+\}$, $\{z_n^0\}$, $\{z_n^-\}$ be the zeros of $f(z)$ in the strip $0\leqq x<\lambda$ whose imaginary parts are positive, zero and negative, respectively. Let

$$u_n^+=\exp(-2\pi i z_n^+/\lambda),\quad u_n^0=\exp(2\pi i z_n^0/\lambda),\quad u_n^-=\exp(2\pi i z_n^-/\lambda).$$

Let $g^+(u)$, $g^0(u)$, $g^-(u)$ be integral functions where zeros are $\{u_n^+\}$, $\{u_n^0\}$, $\{u_n^-\}$, respectively. Then

$$f(z)=e^{h(z)}g^+(e^{-2\pi i z/\lambda})g^0(e^{2\pi i z/\lambda})g^-(e^{2\pi i z/\lambda}),$$

where $h(z)$ is an integral function and $h'(z)$ has the period λ. Among the consequences of this theorem is that the only functions of finite order with a finite number of zeros in a periodic strip are those of order one and finite type.

N. Levinson (Cambridge, Mass.).

Iyengar, K. S. K. A property of integral functions of order less than two with real roots. Ann. of Math. (2) **42**, 823–828 (1941). [MF 5518]

Except for unessential changes, this paper is identical with the author's paper in Proc. Indian Acad. Sci., Sect. A. **12**, 223–229 (1940) [cf. these Rev. **2**, 81]. *G. Pólya.*

Kametani, Shunji. The exceptional values of functions with the set of linear measure zero, of essential singularities. Proc. Imp. Acad. Tokyo **17**, 117–120 (1941). [MF 5390]

Suppose that $f(z)$ is meromorphic in a domain D except on a set E of essential singularities. M. L. Cartwright has proved that, if E is of linear measure zero, then $f(z)$ assumes near each point of E all finite values except perhaps those belonging to a set Ω of plane measure zero. The author gives a more precise estimate of the measure of Ω, and shows that Ω is of h-measure zero (in the Hausdorff sense) if $\lim_{\epsilon\to+0}\int_\epsilon^\tau h(t)dt/t^2<\infty$ for some $\tau>0$, in particular, therefore, that, for any $\epsilon>0$, Ω is a set of $(1+\epsilon)$-dimensional measure zero. *D. C. Spencer* (Cambridge, Mass.).

Landau, Edmund. Ausgewählte Kapitel der Funktionentheorie. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **8**, 23–68 (1940). (German. Russian summary) [MF 5310]

This paper consists of the notes of the lectures delivered by E. Landau at the summer semester of 1931 at Stanford University under the title of "Selected topics from the theory of functions." A great many of the forty-one theorems contained in this work are concerned with the properties of analytic functions $f(z)$, regular for $|z|\leqq 1$, which omit the two values 0 and 1. These results were for the most part obtained by Landau, Hurwitz and Schottky in the period 1904 to 1910. Much of this theory is similar to that contained in the seventh chapter of the second edition of the author's book "Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie" [Berlin, 1929]. The theorem of Bloch [1924] that there is a universal constant P and a point $\gamma=\gamma(f)$ such that the map of $|z|\leqq 1$ by $w=f(z)$ covers the circle $|w-\gamma|<P$ whenever $f(z)$ is regular for $|z|\leqq 1$, $|f'(0)|\geqq 1$ is presented, and together with extensions by Landau and Valiron this topic forms a prominent part of the paper. In relation to the theorem of Bloch some theorems on schlicht functions due to the author and to Dieudonné are included. To carry out the proofs of the principal theorems the classical inequalities of Schwarz and Lindelöf, Rouché's [1862] theorem on the number zeros of $f(z)+g(z)$ as compared to those of $f(z)$, Picard's [1879] theorem that an entire function omitting the values 0 and 1 is a constant, and many others stemming from earlier papers of the author appear with modified proofs. All are beautifully woven into the concise, coherent and logical pattern which characterizes the author's book already mentioned. *M. S. Robertson* (New Brunswick, N. J.).

Kwesselawa, D. Zum Lindelöfschen Prinzip. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 713–718 (1940). (Russian. German translation) [MF 5288]

Loomis, Lynn H. The decomposition of meromorphic functions into rational functions of univalent functions. Trans. Amer. Math. Soc. **50**, 1–14 (1941). [MF 4865]

Given $f(z)$ meromorphic in a simply-connected region R (which may without loss of generality be taken to be the unit circle), the author asks when it is possible to write (1) $f(z)=f_2(f_1(z))$, where $f_2(\zeta)$ is a rational function and $\zeta=f_1(z)$ is regular, bounded and univalent in R. He also asks what restrictions on $f(z)$ correspond to restrictions placed on $f_2(z)$ and $f_1(z)$. The author calls a function $f(z)$ rationally-multivalent at a point z_0 if there is a rational function $w=f_2(z)$ such that $f_2^{-1}(f(z))$ is meromorphic and univalent in a neighborhood of z_0 (neighborhood being defined in a natural way when z_0 is a boundary point of R); $f(z)$ is power-multivalent if $f_2(z)$ can be taken to be a power. The author's fundamental theorem is that a necessary and sufficient condition for (1) is that $f(z)$ is rationally-multivalent at every point of $|z|=1$. From this it follows that (2), if $f(z)$ is also continuous in $|z|\leqq 1$, a necessary and sufficient condition for (1) is that $f(z)$ is power-multivalent at every point of $|z|=1$; and that (3) $f(z)$ is meromorphic in $|z|\leqq 1$ if and only if (1) is possible with $f_1(z)$ regular and univalent in $|z|\leqq 1$. In these results $f_1(z)$ is a mapping function respectively for a general simply-connected region, a simply-connected region with only accessible boundary points and a Jordan region bounded by an analytic curve. A deeper theorem states that a necessary and sufficient condition for (1) with $f_1(z)$ a mapping function for a Jordan region is that for every z_0 on $|z|=1$ there are an integer n and an arc γ of $|z|=1$ containing z_0, such that for some w_0 the limiting values of $[f(z)-w_0]^{1/n}$ as $|z|\to 1$ along γ lie on a Jordan arc. A simpler sufficient condition is that the limiting values of $f(z)$ as $|z|\to 1$ lie on a closed curve with at most a finite number of self-intersections. If $f(z)$ is regular and bounded in $|z|<1$, then (2) and (3) are true with $f_2(\zeta)$ restricted to be either a polynomial or a circular rational function $K\prod_{j=1}^{n}(z-a_j)/(1-\bar{a}_j z)$, $|a_j|<1$. An example is given to show that, if $f(z)$ is merely meromorphic in $|z|<1$, $f_2(\zeta)$ cannot be so restricted. A number of other interesting decomposition theorems are also proved. An interesting auxiliary result is that, if $f(z)$ is meromorphic in $|z|<1$ and if the limiting values of $f(z)$ as $|z|\to 1$ along an arc γ of $|z|=1$ lie on a Jordan arc, then $f(z)$ is continuous in the interior of γ. *R. P. Boas, Jr.* (Durham, N. C.).

Broggi, Ugo. Sulle funzioni regolari nel circolo unità. Boll. Un. Mat. Ital. (2) **3**, 363–366 (1941). [MF 5702]

By a conformal mapping, the author transforms some theorems of Mellin [Ann. Acad. Sci. Fennicae (A) **18** (1922); Doetsch, Theorie und Anwendung der Laplace-Transformation, Berlin, 1937] concerning the identical vanishing of functions analytic in a half-plane into theorems applying to functions analytic in a circle. *R. P. Boas, Jr.*

García Frías, Roque. On the coefficients of functions univalent in the unit circle. Actas Acad. Ci. Lima **4**, 76–85 (1941). (Spanish) [MF 5229]

Let $z+\sum_1^\infty c_{2n+1}z^{2n+1}$ and $z+\sum_1^\infty a_n z^n$ be univalent in the unit circle. Then $|c_9|\leqq 1.164$, $|a_5|\leqq 5.6138$, $|a_6|\leqq 8.0138$, $|a_7|\leqq 11.135$. Proofs by elementary but rather elaborate numerical calculation, as in previous work by Rosenblatt. Use is made of an inequality of M. S. Robertson [Bull. Amer. Math. Soc. **42**, 366–370 (1936)]. *R. P. Boas, Jr.*

Spencer, D. C. On mean one-valent functions. Ann. of Math. (2) **42**, 614–633 (1941). [MF 4961]

Let $f(z)$ be regular in $|z|<1$ with as transform the Riemann domain W. Let $W(R)$ be the area (multiply covered regions being counted multiply) of that portion of W which lies in $|w|\leqq R$. If $W(R)\leqq p\pi R^2$ for all $R>0$, $f(z)$ is said to be mean p-valent. Let $n(r, w)$ be the number of times that $f(z)$ takes the value w in $|z|<r$. Let $n(w)=\lim_{r\to 1}n(r, w)$. Then, if for all $R>0$, uniformly in ψ,

$$\int_0^R n(Re^{i\psi})d(\pi R^2)\leqq p\pi R^2,$$

$f(z)$ is said to be strongly mean p-valent.

In this paper the author continues his study of functions mean 1-valent and shows that such functions possess some of the striking properties of schlicht or univalent functions. For instance, by means of a generalization of the Bieberbach-Faber Flächensatz he obtains $|a_2| \leqq 2|a_1|$ for $f(z)=a_1z+a_2z^2+\cdots$, when $f(z)$ is mean 1-valent. Again, if d denotes the distance of the boundary of the map W from $w=0$, then $d \geqq |a_1|/4$ if $f(z)$ is strongly mean 1-valent, and in this case also $|f(z)| \geqq |a_1| \cdot |z|(1+|z|)^{-2}$, with equality if, and only if, $f(z)=a_1z(1+\gamma z)^{-2}$, $|\gamma|=1$.

The author's methods also yield two results of a more general nature. Suppose $f(z)=a_1z+a_2z^2+\cdots$ is regular for $|z|<1$, that $n(0)=1$, and that $d=\inf_{|z|<1}|f(z)/z|$. Then, if there is a β for which $n(de^{i\beta})=0$, we have $d>|a_1|/13$. If in addition $|f(z)|<M$ in $|z|<1$ and $0<\rho<1$, then $|\arg f| \leqq B \lg\{1/(1-\rho)\}+O(1)$, where $B=(1/\pi)\lg(13M/|a_1|)$. The author points out that some of the results obtained for mean 1-valent functions may be immediately transferred to mean p-valent functions of the form $f(z)=\sum_{n=p}^{\infty}a_nz^n$ and obtains

$$|a_{p+1}| \leqq 2p \cdot |a_p|, \quad d=\inf_{|z|<1}|f(z)/z^p| > |a_p|/7^p, \text{ etc.}$$

M. S. Robertson (New Brunswick, N. J.).

Spencer, D. C. On finitely mean valent functions. Proc. London Math. Soc. (2) **47**, 201–211 (1941). [MF 5402]

A function $f(z)$, regular for $|z|<1$, is said to have valency p if it takes no value w more than p times. More generally, $f(z)$ is said to have mean valency $p>0$ if $W(R) \leqq p\pi R^2$ for all $R>0$, where $W(R)$ denotes the area (regions covered multiply being counted multiply) of that portion of the transform of the unit circle by $w=f(z)$ which lies in $|w| \leqq R$. If $A(\rho)$ denotes the area of the transform of $|z| \leqq \rho<1$ by $f(z)$, $M(\rho, f)$ the maximum modulus of $f(z)$ in $|z|=\rho$, and if $A(\rho) \leqq p\pi M^2(\rho, f)$, $f(z)$ is said to have weak mean valency p.

The author generalizes known results of univalent and p-valent functions to functions having mean valency p. For $\lambda>0$, $\rho<1$ define

$$M_\lambda(\rho, f)=\left((2\pi)^{-1}\int_{-\pi}^{\pi}|f(\rho e^{i\theta})|^\lambda d\theta\right)^{1/\lambda}.$$

If $f(z)$ has mean valency $p \geqq 1$, $f(0)=0$, $\lambda>0$, then

$$M_\lambda{}^\lambda(\rho, f) \leqq p\Lambda\int_0^\rho M^\lambda(r, f)dr/r,$$

where

$$\Lambda=\begin{cases}\lambda, & 0<\lambda \leqq 2,\\ \lambda^2/2, & \lambda>2.\end{cases}$$

If in addition $\alpha>0$, and $M(\rho, f) \leqq C(1-\rho)^{-\alpha}$, then, for $\lambda>1/\alpha$,

$$M_\lambda(\rho, f) \leqq CK(\rho, \lambda)(1-\rho)^{-\alpha+1/\lambda}.$$

If $\alpha \geqq 0$, $\lambda>(\alpha+\frac{1}{2})^{-1}$ and $M(\rho, f) \leqq C(1-\rho)^{-\alpha}$, then

$$M_\lambda(\rho, f') \leqq CK(\rho, \lambda, \alpha)(1-\rho)^{-\alpha-1+1/\lambda}.$$

An example is given to show that the condition $\lambda>(\alpha+\frac{1}{2})^{-1}$ cannot be relaxed to the extent of $\lambda>(\alpha+1)^{-1}$. Moreover, neither of these results remain true for functions of weak mean valency p.

As an immediate consequence of the above results it follows by a well-known method that, if $f(z)=\sum a_nz^n$ has mean valency $p \geqq 1$, and $M(\rho, f) \leqq C(1-\rho)^{-\alpha}$, $\alpha>\frac{1}{2}$, then $|a_n| \leqq CK(p, \alpha)n^{\alpha-1}$. In particular, if

$$f_k(z)=a_1z+a_{k+1}z^{k+1}+a_{2k+1}z^{2k+1}+\cdots$$

has valency $p \geqq 1$, then $|a_n|<K(\rho, k)\mu_q n^{2p/k-1}$ for $k<4p$, where $\mu_q=\max\{|a_1|, |a_{k+1}|, \cdots, |a_{(q-1)k+1}|\}$ and $q(\leqq p)$ is the number of zeros of $f_k(z)$ in $|z|<1$. This last inequality for $k=1$ is due to M. Biernacki [C. R. Acad. Sci. Paris 204, 449–451 (1936)], and for $1<k<4p$, in case $f_k(z)$ has no zeros other than the origin, to the reviewer [Bull. Amer. Math. Soc. **44**, 282–285 (1938)]. The method of proof of the main results depends upon the Hardy-Stein identity, the Prawitz identity and the formulation of the hypothesis of mean valency p as $\int_0^R p(R)d\pi R^2 \leqq p\pi R^2$ for every $\rho<1$, where $p(R)=(2\pi)^{-1}\int_{-\pi}^{\pi}N(Re^{i\psi})d\psi$ and $N(Re^{i\psi})$ is the number of times the value $Re^{i\psi}$ is taken by $f(z)$ in $|z| \leqq \rho$. The results above hold true for $p>0$ in place of $p \geqq 1$ if $W(R) \leqq p\pi R^2$ for all $R>R_0$ provided the constant factors in the upper bounds depend on the additional parameter R_0.

M. S. Robertson (New Brunswick, N. J.).

Robertson, M. S. The partial sums of multivalently star-like functions. Ann. of Math. (2) **42**, 829–838 (1941). [MF 5519]

Suppose that, for $|z|<1$,

$$f(z)=z^p+a_{p+1}z^{p+1}+\cdots+a_nz^n+\cdots$$

is regular and p-valent, and that in addition

$$\Re[zf'(z)/f(z)]>0;$$

such a function is said to be multivalently star-like of order p. In an earlier paper [M. S. Robertson, Ann. of Math. (2) **37**, 374–408 (1936)] the author showed that for $n>n_0$ the nth partial sum of the power series of a univalently star-like function is univalently star-like for $|z|<1-cn^{-1}\log n$, where c is a constant which he asserted is 2. The author begins by correcting a slight mistake in his estimation of c, and points out that the corrected computation yields as estimate for c the value 4 (instead of 2). He then generalizes this result, and proves that for a multivalently star-like function of order p the nth partial sum is multivalently star-like for $|z|<1-(2p+2)n^{-1}\log n$, provided that $n>n_0(p)$.

D. C. Spencer (Cambridge, Mass.).

Kobori, Akira. Sur les fonctions multivalentes. Proc. Phys.-Math. Soc. Japan (3) **23**, 423–431 (1941). [MF 5406]

The author considers p-valent functions f of the form $f(z)=\{h(z)\}^p$, where $h(z)$ is regular and schlicht for $|z|<1$ and $h(0)=0$, $h'(0)=1$, and finds bounds for the coefficients and modulus of f from well-known results for h.

D. C. Spencer (Cambridge, Mass.).

Wolff, J. Séries se rapportant aux fonctions holomorphes bornées. Nederl. Akad. Wetensch., Proc. **44**, 619–624 (1941). [MF 4981]

Suppose that $w=u+iv=f(z)=f(x+iy)$ is regular, and that $u>0$, in the half-plane $D(x>0)$. Let S denote a sequence of points $\alpha_n=a_n+ib_n$, $n=1, 2, \cdots$, of D satisfying: (i) $|a_{n+1}| \geqq c|a_n|$, $n=1, 2, \cdots$, for fixed $c>1$; (ii) $-\pi/2+\epsilon \leqq \arg \alpha_n \leqq \pi/2-\epsilon$, $n=1, 2, \cdots$, for fixed $\epsilon>0$. The author proves for sequences S that

$$(1) \qquad \sum_{n=1}^{\infty}|\Im(f(\alpha_n)/\alpha_n)|<\infty.$$

(1) is shown to be equivalent to the convergence of

$$(2) \qquad \sum_{n=1}^{\infty} |\arg (F(\alpha_n)/\alpha_n)|,$$

where $F=z+f$; and convergence of (2) is deduced from Fatou's theorem which states that

$$\int_{-\infty}^{+\infty} \left|\arg \frac{F(ti)}{ti}\right| \cdot \frac{dt}{1+|t|} < \infty,$$

where $F(ti)=\lim_{x\to 0} F(x+ti)$ is defined almost everywhere. Several other results of a similar type are given.

D. C. Spencer (Cambridge, Mass.).

Wolff, J. Sur l'itération d'une représentation conforme. III. Nederl. Akad. Wetensch., Proc. **44**, 308–309 (1941). [MF 5013]

[For the preceding note, cf. the same Proc. **44**, 195–197 (1941); these Rev. **2**, 358.] Suppose that $z_1(z)$, $z_1(1)=1$, maps conformally the half-plane $D(\Re z>0)$ onto a domain $D_1 \subset D$, the frontier f of D_1 being a Jordan curve interior to D and extending to infinity in the two directions $\arg z=\pm\pi/2$; and suppose that the angular derivative λ of $z_1(z)$ at infinity is positive. Let $z_n(z)=z_{n-1}\{z_1(z)\}$, and $D_n=z_n(D)$, for $n\geqq 2$, and let $N=\prod_{n=1}^{\infty} D_n$. In the preceding note the author has shown that N contains a domain G of angular width π at infinity. Here he derives from $z_1(z)$ a function which represents G conformally on D.

D. C. Spencer (Cambridge, Mass.).

Vladimirsky, Serge. Sur la représentation conforme des domaines limités intérieurement par des segments rectilignes et arcs circulaires. C. R. Acad. Sci. Paris **212**, 379–382 (1941). [MF 4910]

The author gives, by means of hyperelliptic integrals, a generalization of the solution of J. Hodgkinson and E. Poole [Proc. London Math. Soc. (2) **23**, 396–422 (1924)] of a problem of the conformal representation of doubly-connected regions bounded by rectangular segments or circular arcs, extending this solution to the case where the connectivity of the region exceeds two. *M. H. Heins.*

Yang, Ou Tchen. Sur la détermination des expressions limitatives exactes dans certains théorèmes de la théorie des fonctions analytiques. C. R. Acad. Sci. Paris **211**, 59–61 (1940). [MF 5335]

A fundamental theorem of Lindelöf compares the Green's functions for two Riemann surfaces S and S' when S' is carried by S [see G. Julia, Principes geometriques d'analyse, part 2, pp. 68–90, 1932]. The author announces the result that the theorem still holds when the restriction "carried by" is replaced by the weaker "quasi-carried by." The surface S' is said to be carried by S provided S' is a covering surface of S such that every closed curve on S' is projected on a closed curve on S; and S' is said to be quasi-carried by S if (1) every closed curve on S' containing no branch point in its interior is projected on a closed curve on S and (2) every small closed curve on S' about a branch point P' is projected on a curve, closed or not, which contains a small loop about the point P homologous to P'. The present result and methods allow the extension of certain function-theoretic results from exceptional values of rank r_i to exceptional values of weight not less than $1-(1/r_i)$. In particular, the author announces the determination of the exact values of the limiting expressions which figure in generalizations [see R. M. Robinson, Duke Math. J. **5**, 118–132 (1939)] of the theorems of Picard, Montel, Landau and Schottky.

E. F. Beckenbach (Ann Arbor, Mich.).

Kaila, Esko. Über die Ränderzuordnung bei konformer Abbildung von mehrfach zusammenhängenden Gebieten. Ann. Acad. Sci. Fennicae (A) **55**, no. 9, 63 pp. (1940). [MF 3883]

Let G be a multiply connected simple region with at least three boundary points, and let $w=w(z)$ map G conformally onto the interior of the circle $C:|w|=1$. The function $w(z)$ is multi-valued (the map is one-to-infinity) and the various branches of $w(z)$ are connected by the linear substitutions of a group $\{S\}$, each of which leaves C invariant. Two points (or sets) in the w-plane are equivalent under $\{S\}$ if one can be obtained from the other by a substitution of $\{S\}$. The paper deals with the behavior of the mapping function at an accessible boundary point. Its principal object is the exposition of results of Koebe, Bieberbach, Julia and Nevanlinna, with complete and uniform proofs, based upon Nevanlinna's definition of an accessible boundary point. Any path $L: z=z(t)$, $0\leqq t\leqq 1$, which lies in G except for the end point $E(t=1)$, is said to define an accessible boundary point at E; two paths L' and L'' define the same boundary point (a) if they have the same end point E, and (b) if there exist sequences of points P_n' on L', P_n'' on L'', whose parameter values increase with n and which approach E as $n\to\infty$, such that the following condition is satisfied: P_n' and P_n'' may be connected by arcs in G which converge to E as $n\to\infty$, and for a certain $n=n_0$ and every $\rho>0$, the closed curve $P'_{n_0}P'_{n_0+\rho}P''_{n_0+\rho}P''_{n_0}P'_{n_0}$ can be contracted to a point in G without ever meeting the boundary. [The author remarks that this definition seems better suited for the problem than the one suggested by Bieberbach and later on extended by O. Teichmüller [Deutsche Math. **4**, 455–461 (1939); cf. these Rev. **1**, 8]. The principal results proved in the paper are: (a) To every accessible boundary point E of G there corresponds a set of points $\{\alpha\}$ on C, equivalent under $\{S\}$, in the sense that, whenever $z\to E$ along a path L, the image $w=w(z)$ describes a path in C ending at a point α of this set; conversely, if $w\to\alpha$ along a non-tangential path Λ then $z\to E$ along a path L, the image of Λ. The totality of sets $\{\alpha\}$ on C corresponding to all accessible boundary points of G forms a set of linear measure 2π or 0, according as G is of bounded type or not. [G is of bounded type if the inverse function $z=z(w)$ of $w(z)$ can be represented as a quotient of two bounded analytic functions.] (b) Each of the points $\{\alpha\}$ on C which correspond to an isolated boundary point z_0 is a fixed point of some parabolic substitution of $\{S\}$, and conversely, any fixed point of a parabolic substitution of $\{S\}$ belongs to some set $\{\alpha\}$ which corresponds to an isolated boundary point. When z describes a closed contour about such a point z_0, then any branch of $w(z)$ undergoes a parabolic transformation of $\{S\}$ whose fixed point is the image of z_0 by means of this branch. (c) Any open "free" Jordan arc β of the boundary of G is mapped onto a set of open, non-overlapping arcs $\{\gamma\}$ of C, equivalent under $\{S\}$, in the sense that any given branch of $w(z)$ is continuous on β and maps it in a one-to-one manner onto some γ. The end points of each γ are fixed points of some hyperbolic substitution of $\{S\}$. Of particular interest among the lemmas of the paper are extensions of two classical inequalities of Lindelöf [Acta Soc. Sci. Fen-

nicae **46**, no. 4 (1915)] whose proofs are based on the use of harmonic measure and seem simpler than those of the original inequalities. *S. E. Warschawski.*

Heins, Maurice H. A generalization of the Aumann-Carathéodory "Starrheitssatz." Duke Math. J. **8**, 312–316 (1941). [MF 4595]

The author proves the following theorem: If F_w is a Riemann surface which is not simply connected and which has the properties that $\bar{F}_w$, the universal covering surface, is of hyperbolic type and that the fundamental group G associated with F_w is not cyclic, then the identical map of F_w onto itself can never be expressed as the limit of a sequence $W_n(w) \not\equiv w$ of single valued functions defined and analytic in F_w and such that $w \in F_w$ implies $W_n(w) \in F_w$.

The "Starrheitssatz" of Aumann-Carathéodory asserts the existence of an upper bound $\Omega(\zeta, F_w) < 1$ for $W'(\zeta)$ among all functions $W(w)$ of the type described in the theorem and such that $W(\zeta) = \zeta$, and follows as an immediate corollary of the above theorem. In the "Starrheitssatz" theorem, however, F_w is required to be planar. In addition, other corollaries include the theorem that the number of (1, 1) conformal maps of a plane region bounded by more than two Jordan curves onto itself is finite. The author also obtains the best possible results when the conditions on F_w of the theorem are not satisfied, as in the case of an annular ring. In this case the theorem as stated above is not valid, but is so if we require that $W_n(w)$ be not a conformal map of F onto itself. In the case $\bar{F}_w$ is not hyperbolic, neither result is valid. *J. W. Green.*

Dufresnoy, Jacques. Sur la théorie d'Ahlfors des surfaces de Riemann. C. R. Acad. Sci. Paris **212**, 595–598 (1941). [MF 4922]

The following theorem based on a theorem of Ahlfors [Acta Math. **65**, 157–194 (1935), in particular p. 168] is proved in outline: If $w = f(z)$ is analytic for $|z| < R (< \infty)$ and if one of the four following conditions is fulfilled: (a) $w = f(z)$ has no "island" over two finite domains D_i $(i = 1, 2)$, (b) $w = f(z)$ omits two finite values a_i $(i = 1, 2)$, (c) $w = f(z)$ has no simple "island" over three finite domains $D_i (i = 1, 2, 3)$, (d) $w = f(z)$ has three finite values $a_i (i = 1, 2, 3)$ completely ramified, then

$$\log |f(z)| < \frac{2R}{R-r}\left[h^2 \log \frac{R}{R-r} + \log (1 + |f(0)|^2)\right]$$

for $|z| \leqq r < R$, h being a constant depending only on the domains D_i (or the points a_i). *M. H. Heins.*

Dufresnoy, Jacques. Sur une nouvelle démonstration d'un théorème d'Ahlfors. C. R. Acad. Sci. Paris **212**, 662–665 (1941). [MF 4925]

Let Σ_0 denote the Riemann sphere, let $D_i (i = 1, 2, \cdots, q)$ denote q disjoint sets which are the closures of simply-connected regions on Σ_0 or isolated points, and finally let Σ_0^* denote Σ_0 with the sets D_i deleted; $4\pi S_0$ is the area of Σ_0^*. The theorem "If Σ is a bounded covering surface of Σ_0^* with area $4\pi S_0 S$ and relative boundary length L, then the characteristic ρ of Σ satisfies $\rho^+ \geqq (q-2)(S - 2kL)$, where k is a constant depending only on the D_i" leads to the following theorem of Ahlfors [Acta Math. **65**, 157–194 (1935)]: Given $q (\geqq 3)$ simply-connected disjoint domains D_i on the Riemann sphere Σ_0, one can determine a constant h such that every simply-connected bounded covering surface Σ of Σ_0 has "islands," on the domains D_i, the sum of whose simple multiplicities $\sum p(D_i)$ satisfies

$$\sum_{i=1}^{q} p(D_i) \geqq (q-2)(S - hL),$$

where $4\pi S$ is the area of Σ and L the length of its contour. The advantage of this approach is that h can be appraised sharply for simple types of D_i. *M. H. Heins.*

Dufresnoy, Jacques. Sur certaines propriétés nouvelles des fonctions algébroïdes. C. R. Acad. Sci. Paris **212**, 746–749 (1941). [MF 4930]

Ahlfors' theorem [see preceding review] is extended to multiply-connected Riemann surfaces and this extension is applied to the theory of algebroid functions.

M. H. Heins (Princeton, N. J.).

Dufresnoy, Jacques. Sur une propriété de la fonction de croissance $T(r)$ d'un système de fonctions holomorphes. C. R. Acad. Sci. Paris **211**, 536–538 (1940). [MF 5363]

Two geometric interpretations are given to the quantity $t(r)$ of the theory of meromorphic curves [H. and J. Weyl, Ann. of Math. (2) **39**, 516–538 (1938); in particular, p. 523] which are analogues of the geometric interpretation of $t(r)$ in the case of a single meromorphic function.

M. H. Heins (Princeton, N. J.).

Dufresnoy, Jacques. Sur les théorèmes fondamentaux de la théorie des courbes méromorphes. C. R. Acad. Sci. Paris **211**, 628–631 (1940). [MF 5367]

A new proof of the third fundamental theorem of the theory of meromorphic curves [H. and J. Weyl, Ann. of Math. (2) **39**, 516–538 (1938); in particular, p. 533 ff.] is given, and an estimate for the compensating term Ω is obtained. *M. H. Heins* (Princeton, N. J.).

Weyl, Joachim. Analytic curves. Ann. of Math. (2) **42**, 371–408 (1941). [MF 4290]

This paper is an extension of an earlier one, which we call C_1, by H. and J. Weyl [Ann. of Math. (2) **39**, 516–538 (1938)], in which the authors generalized R. Nevanlinna's theory of meromorphic functions (in the form developed by Ahlfors) to meromorphic curves $\mathfrak{C}$ in the complex k-dimensional space with the homogeneous coördinates x_j $(j = 0, 1, \cdots, k)$, $\mathfrak{C}$ being defined by setting up x_j/x_0 $(j = 1, 2, \cdots, k)$ as meromorphic functions of a complex variable z ranging over the whole z-plane, excluding $z = \infty$. Here the author supposes that x_j/x_0 $(j = 1, 2, \cdots, k)$ are functions of rational character on an arbitrary Riemann surface $\mathfrak{F}$ (the meromorphic curves of C_1 appearing as a special case), and develops the first and second main theorems. Moreover, the defect relations are established in the following two cases (in addition to the case where $\mathfrak{F}$ is the finite z-plane): (i) where $\mathfrak{F}$ is an n-sheeted unbounded covering surface of the finite z-plane; (ii) where $\mathfrak{F}$ is the doubly punctured z-sphere. The validity of these relations is established independently of the restriction imposed in C_1 that the exceptional points satisfy no accidental linear relations. The concepts connected with the k-dimensional projective space introduced in C_1 are again used. In particular, the distance $\|\alpha x\|$ between a point x and a plane $(\alpha y) = \sum \alpha_j y_j = 0$ is given by

$$\|\alpha x\|^2 = |\sum \alpha_j x_j|^2 / \sum |\alpha_j|^2 \cdot \sum |x_j|^2.$$

Let G_0 be a fixed, compact part of $\mathfrak{F}$, and let G be a region containing G_0 in its interior, the closure of G being compact. It is assumed that the boundaries Γ_0 and Γ of G_0 and G,

respectively, consist of a finite number of simple closed curves with continuously varying tangents. A potential ϕ is defined by the conditions that it is a positive constant K on G_0, zero in $\mathfrak{F}-G$; ϕ is then harmonic in $G-G_0$, and continuous over the whole of $\mathfrak{F}$. Writing $d\sigma=-(1/2\pi)(\partial\phi/\partial n)ds$ on Γ_0, $d\sigma=+(1/2\pi)(\partial\phi/\partial n)ds$ on Γ, where n is the normal directed toward the interior of $\mathfrak{F}$ and ds is an element of arc length, the author normalizes ϕ by supposing that $\int_\Gamma d\sigma=\int_{\Gamma_0}d\sigma=1$. If F is a function meromorphic on $\mathfrak{F}$, an application of Green's theorem yields the generalized Jensen identity

$$(1)\quad \int_\Gamma \log|F|d\sigma-\int_{\Gamma_0}\log|F|d\sigma=\sum_{p_0}\phi(p_0)-\sum_{p_\infty}\phi(p_\infty),$$

where the sums on the right are extended over all zeros p_0 and poles p_∞, respectively, of F on $\mathfrak{F}$, multiplicity counting (but only the finite number of them contained in G actually contribute to these sums). Taking $F=(\alpha x)/(\beta x)$ in (1), and writing $N(G;\alpha)=\sum\phi(p_0)$,

$$\int_\Gamma \log\|\alpha x\|^{-1}d\sigma=m(G;\alpha),\qquad \int_{\Gamma_0}\log\|\alpha x\|^{-1}d\sigma=m^0(G;\alpha),$$

the combination $T(G)=N(G;\alpha)+m(G;\alpha)-m^0(G;\alpha)$ is seen to be independent of the particular plane α (the first main theorem). Furthermore, it is shown that, if $G_1\subset G_2$, $T(G_1)\leqq T(G_2)$. Exhausting $\mathfrak{F}$ by a sequence of regions G_r, $G_r\subset G_{r'}$ if $r'>r$, $T(r)=T(G_r)$ is defined to be the order of the curve $\mathfrak{C}$. The second main theorem is obtained from (1) in a manner analogous to that of C_1. The most interesting results of the paper appear in connection with the defect relations, but unfortunately lack of space makes it undesirable for the reviewer to state these results here.

D. C. Spencer (Cambridge, Mass.).

Montel, Paul. Sur les points où une fonction analytique est égale à une fonction algébrique. C. R. Acad. Sci. Paris 211, 217–220 (1940). [MF 5345]

Let, for $m\geqq 3$, the equation

$$F(x,y)=A_0(x)y^m+A_1(x)y^{m-1}+\cdots+A_{m-1}(x)y+A_m(x)=0$$

define one or more algebraic functions $y(x)$, the total number of whose determinations is not less than three. The author announces the following results. The functions $f(x)$, meromorphic and different from $y(x)$ at each point of a domain (D), form a normal family in the interior of this domain. If the functions are holomorphic it is sufficient to take $m\geqq 2$. The numbers three or two in these propositions cannot be replaced by smaller integers. A function meromorphic in the neighbourhood of an isolated essential singularity takes infinitely often the values of every algebraic function admitting at least three determinations. This theorem contains the theorem of Picard when $F(x,y)\equiv(y-a)(y-b)(y-c)$, where a, b, c are different constants. For holomorphic functions two determinations of the algebraic function suffice.

A polynomial is said to be exceptional relative to $F(x,y)$ when it is either always different from $y(x)$ in the finite part of the plane or when it coincides with a branch of the algebraic function. The maximum number of exceptional polynomials relative to $F(x,y)$ is equal to m except in the unique case where $F(x,y)$ is a polynomial with constant coefficients with respect to the variable $y-P(x)$, $P(x)$ being a polynomial. In this last case, every polynomial of the form $P(x)+$const. is exceptional.

$F(x,y)$ is said to be an ordinary or a singular polynomial depending upon whether it admits a finite (including none at all) or an infinity of exceptional polynomials. The following is a generalization of a theorem of Landau. Suppose $f(x)$ holomorphic for $|x|<R$ and in this circle different from the algebraic function defined by the polynomial $F(x,y)$. If $F(x,y)$ is an ordinary polynomial and if $f(0)=a_0$ is a fixed value different from the constant coefficients of the exceptional polynomials, the numbers R have a finite superior limit $R_0(a_0)$ depending only upon a_0. If $F(x,y)$ is a singular polynomial and if $f(0)=a_0$, $f'(0)=a_1$ are fixed numbers of which a_1 differs from $P'(0)$ relative to the exceptional polynomials $P(x)+$const., the numbers R have a finite superior limit $R_0(a_0,a_1)$ depending only upon a_0 and a_1. The author gives several special applications of this theorem.

The functions $f(x)$, holomorphic in the unit circle, taking at the origin a fixed or bounded value, and different from an algebraic function with at least two determinations, are uniformly bounded in every interior circle. This is an extension of the theorem of Schottky. With a definition of an exceptional rational function analogous to that of an exceptional polynomial, corresponding theorems are obtained for functions $f(x)$ meromorphic for $|x|<R$.

M. S. Robertson (New Brunswick, N. J.).

Montel, Paul. Sur les fonctions analytiques égales à une fonction algébrique en un nombre borné de points. C. R. Acad. Sci. Paris 211, 370–374 (1940). [MF 5354]

Let the algebraic function $y(x)$ be defined by the equation

$$F(x,y)\equiv A_0(x)y^m+A_1(x)y^{m-1}+\cdots+A_{m-1}(x)y+A_m(x)=0,$$

in which $A_i(x)$, $i=0,1,\cdots,m$, are polynomials, $A_0(x)$ being of degree a_0, and where m is an integer greater than two. The functions $f(x)$ meromorphic in a domain (D), where they take at most p times the values of an algebraic function $y(x)$ with at least three determinations, form a quasi-normal family of finite total order. If the functions $f(x)$ are holomorphic in (D) and p is less than $m+a_0$ the family is normal.

A polynomial $P(x)$ is said to be exceptional of order p relative to $F(x,y)$ if $P(x)$ equals $y(x)$ p times at most or coincides with a branch of this function. Let

$$f(x)=a_0+a_1x+\cdots+a_nx^n+\cdots$$

be holomorphic for $|x|<R$ and let $f(x)$ equal at most p times the algebraic function, with at least two determinations, defined by the polynomial $F(x,y)$. If $F(x,y)$ admits no exceptional polynomial of order p, the number R is bounded by a fixed number R_0 which depends only on $F(x,y)$. If $F(x,y)$ admits a finite number of exceptional polynomials of order p and if the coefficients $a_0, a_1, \cdots, a_q$ are fixed, a_0 being different from the values at the origin $P(0)$ of the exceptional polynomials, one has $R\leqq R_0(a_0,a_1,\cdots,a_q)$, R_0 depending only upon $F(x,y)$ and the values $a_0, a_1, \cdots, a_q$. If $F(x,y)$ admits an infinity of exceptional polynomials $P(x)$ of order p and if the coefficients $a_0, a_1, \cdots, a_q, a_{q+1}$ are fixed, a_{q+1} being different from the values $P^{(q+1)}(0)/(q+1)!$, finite in number, one has $R\leqq R_0(a_0,a_1,\cdots,a_q,a_{q+1})$, R_0 depending only upon $F(x,y)$ and the values $a_0, a_1, \cdots, a_q, a_{q+1}$.

A rational fraction is said to be exceptional of order p relative to $F(x,y)$ if the fraction equals $y(x)$ at p points at most or if the fraction is identical with one of the branches

of $y(x)$. The author obtains a theorem for functions $f(x)=a_0+a_1x+\cdots$ meromorphic for $|x|<R$ analogous to the theorem above, where now the exceptional polynomials are replaced by exceptional rational fractions, and where the algebraic function $y(x)$ has now at least three determinations. *M. S. Robertson* (New Brunswick, N. J.).

Valiron, Georges. Sur le domaine couvert par les valeurs d'une fonction algébroïde finie. Bull. Sci. Math. (2) **64**, 199–206 (1940). [MF 5233]

In a previous paper [Proc. Phys.-Math. Soc. Japan (3) **12**, 64 (1930)], the author has investigated finite algebroid functions $u(z)$ of order ν in $|z|<1$; that is, functions $u(z)$ defined by

$$u^\nu+A_{\nu-1}(z)u^{\nu-1}+\cdots+A_0(z)=0,$$

where the $A_j(z)$ are holomorphic in $|z|<1$. The results obtained there suggest the following generalization of Bloch's theorem. If $u(z)$ is a finite algebroid of order ν in $|z|<1$, if ρ is a constant, $0<\rho<1$, and if for one of the values $u(0)$ and a value $u(z_0)$ obtained from $u(0)$ by describing $|z|\leqq|z_0|=\rho$ we have

$$|u(z_0)-u(0)|>1, \tag{1}$$

then the Riemannian domain described by $u(z)$ for $|z|<1$ contains a circle, not necessarily of a single sheet, of radius superior to some positive absolute constant $B(\nu,\rho)$. The theorem is proved only for $\nu=2$. Further, for $\nu=2$, it is shown by an example that the theorem is not valid when the phrase "not necessarily of a single sheet" is replaced by "of a single sheet." In the above theorem the condition (1) replaces the condition relative to the derivative at the origin which figures in Bloch's theorem but which would here be without sense. *E. F. Beckenbach.*

Moriya, Mikao. Algebraische Funktionenkörper und Riemannsche Flächen. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9**, 209–245 (1941). [MF 5400]

This paper is an attempt to modernize some chapters of H. Weyl's book "Die Idee der Riemannschen Fläche" [Leipzig-Berlin, 1923]. Let $K=k(x,y)$ be a field of algebraic functions of one variable over the field of all complex numbers k. The author shows anew that the set of valuations on K (trivial on k) is a topological space which is homeomorphic to the absolute Riemann surface of K. Having established this connection the author proceeds to develop the well-known theory of finite algebraic covering surfaces. The proofs involving the fundamental and Betti groups are of analytic nature. Occasionally results of the arithmetic theory of function fields are used. In many instances it is necessary to consult Weyl's book for the understanding of essential details. *O. F. G. Schilling.*

Francotte, R. Sur un moyen élémentaire d'établir les formules d'addition de la fonction elliptique $F(k,\phi)$. Mathesis **54**, 110–114 (1940). [MF 5152]

The author derives the addition theorem for the Jacobian elliptic function $\mathrm{sn}(u,k)$. The method used is essentially due to Euler [Calculus Infinitesimalis, vol. 1, chaps. V, VI], and depends on the integration of the differential equation

$$\frac{dx}{((1-x^2)(1-k^2x^2))^{\frac{1}{2}}}+\frac{dy}{((1-y^2)(1-k^2y^2))^{\frac{1}{2}}}=0.$$

M. A. Basoco (Lincoln, Neb.).

van Veen, S. C. Die Entwicklung des vollständigen elliptischen Integrales erster Art in der Nähe von $k=1$. I. Nederl. Akad. Wetensch., Proc. **44**, 606–613 (1941). [MF 4979]

Proof that in the neighborhood of $k=1$ the complete elliptic integral of the first kind can be expanded in the series

$$K(k)=\int_0^{\pi/2}\frac{d\varphi}{(1-k^2\sin^2\varphi)^{\frac{1}{2}}}$$
$$=\frac{2}{\pi(1+k)}\log\left[\frac{2^{\frac{1}{2}}+(1+k)^{\frac{1}{2}}}{(1-k)^{\frac{1}{2}}}\right]\sum_{n=0}^{\infty}\left[\frac{\Gamma(n+\frac{1}{2})}{n!}\right]^2\left(\frac{1-k}{1+k}\right)^{2n}$$
$$+\left(\frac{2}{1+k}\right)^{\frac{1}{2}}\sum_{n=1}^{\infty}(-1)^nA_n\left(\frac{1-k}{1+k}\right)^n,$$

where

$$A_n=\frac{1}{2\pi^{\frac{1}{2}}}\sum_{m=1}^{[(n+1)/2]}\left[\frac{\Gamma(n-m+\frac{3}{4})}{(n-m+1)!}\right]^2\cdot\frac{(n-2m+1)!}{\Gamma(n-2m+\frac{5}{4})}.$$

This expansion is valid for complex values of k in the right half-plane $\Re(k)>0$ when a cut is made from 1 to $+\infty$. *M. C. Gray* (New York, N. Y.).

van Veen, S. C. Annäherungsformeln für das vollständige elliptische Integral erster Art in der Nähe von $k=1$. II. Nederl. Akad. Wetensch., Proc. **44**, 614–618 (1941). [MF 4980]

This is a continuation of the paper reviewed above. The author shows that for real values of k, $0<k<1$, upper and lower bounds of $K(k)$ are:

$$\frac{1}{k^{\frac{1}{2}}}\log\left[\frac{(2k)^{\frac{1}{2}}+(1+k)^{\frac{1}{2}}}{(1-k)^{\frac{1}{2}}}\right]<K(k)<\frac{2}{1+k}\log\left[\frac{2^{\frac{1}{2}}+(1+k)^{\frac{1}{2}}}{(1-k)^{\frac{1}{2}}}\right].$$

For values of k in the neighborhood of $k=1$ the arithmetic mean of these approximations differs from the true value by at most $((1-k)^2/128)((2)^{5/2}/(1+k))$. *M. C. Gray.*

Krygowsky, Zdislas. Sur les intégrales hyperelliptiques canoniques de seconde espèce. Bull. Sci. Math. (2) **64**, 216–225 (1940). [MF 5235]

After a restatement of Weber's formulas for the hyperelliptic integrals ζ_1 and ζ_2 [see J. Reine Angew. Math. **82**, 137 (1877)] with moduli of periodicity 0, 0, $-4\pi i$, 0 and 0, 0, 0, $-4\pi i$, respectively, in terms of the two integrals w_1 and w_2 of the first kind and two, e_1 and e_2, of the second kind, the author introduces two new integrals η_1 and η_2 whose moduli of periodicity are respectively $-4\pi i$, 0, 0, 0 and 0, $-4\pi i$, 0, 0. He gives simple rules for setting up these integrals and also two methods for determining the value of the determinant formed by the moduli of periodicity of w_1, w_2, e_1 and e_2. The work is purely formal. *A. Dresden.*

Boev, G. Über fastautomorphe Funktionen erster Art. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 731–734 (1941). [MF 5239]

By replacing the group of translations by the group of linear fractional transformations having a fixed circle, the author generalizes almost periodic functions to almost automorphic functions. Since it is necessary to deal with meromorphic instead of holomorphic functions, a spherical distance $|w_1,w_2|$ is introduced in the range so that functions with poles may be uniformly approximated. Similarly the Lobachevskian distance $L(z_1,z_2)$ is used to provide an invariant distance for the domain of the functions.

Definition: A function $f(z)$ meromorphic in the unit circle Γ will be called almost automorphic of the first kind if to an arbitrary $\epsilon>0$ there corresponds a set of linear fractional transformations $T_k(z)$ which take the interior of Γ into itself and such that $T_k(0)$ is L-relatively dense in Γ and $|f(T_k(z)), f(z)|<\epsilon$ for all k and all z in Γ. The theorem analogous to the Bochner compactness theorem is proved and thus a simpler alternative definition is obtained. It is shown by an example that non-trivial almost automorphic functions exist. *R. H. Cameron* (Cambridge, Mass.).

Boev, G. Über fastautomorphe Funktionen erster Art. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 837–839 (1941). [MF 5511]

The properties of almost automorphic functions developed in this paper are analogous either to those of almost periodic functions or to those of automorphic functions or both. The boundedness and uniform continuity of almost periodic functions have their counterparts, though of course with modification necessitated by the poles of the functions. Many translation properties of almost periodic functions carry over just as one would expect. Properties of the zeros and convergence of series involving them approximate the properties of automorphic functions. *R. H. Cameron.*

Bergman, Stefan. The method of the minimum integral and analytic continuation of functions of complex variables. Proc. Nat. Acad. Sci. U. S. A. **27**, 328–332 (1941). [MF 4585]

Ist $K_{\mathfrak{B}}(z_1, z_2; \bar{t}_1, \bar{t}_2) \equiv K_{\mathfrak{B}}(z; \bar{t})$ die Kernfunktion eines Bereiches $\mathfrak{B}$, so ist bekanntlich $[K_{\mathfrak{B}}(t; \bar{t})]^{-1}$ das Minimum der Volumenintegrale $\int\int f \bar{f} d\omega$, wobei f alle in $\mathfrak{B}$ quadratintegrierbaren Funktionen mit $f(t)=1$ durchläuft. Ist $\mathfrak{G} \subset \mathfrak{S}$, so gilt: $[K_{\mathfrak{G}}(t; \bar{t})]^{-1} \leqq [K_{\mathfrak{S}}(t; \bar{t})]^{-1}$. Diese "Methode der Minimalintegrale" ist für die Funktionentheorie mehrerer komplexen Veränderlichen ein gewisser Ersatz des Schwarzschen Lemmas. [Zu vorstehenden Begriffen und Sätzen, vgl. Bergmann, Sur les fonctions orthogonales de plusieurs variables complexes, etc., New York, 1941; diese Rev. **2**, 359]. Ist $\mathfrak{L}^2(z_k = z_k(u_1, u_2); 0 \leqq u_1, u_2 \leqq 1)$ ein in $\mathfrak{B}$ liegendes Flächenstück, so wird

$$B_{\mathfrak{B}}(\mathfrak{L}^2) = \int_0^1 \int_0^1 (K_{\mathfrak{B}}(z, \bar{z}))^{\frac{1}{2}} \left| \frac{D(z_1, z_2)}{D(u_1, u_2)} \right| du_1 du_2$$

als nichteuklidischer B-Inhalt des Flächenstückes $\mathfrak{L}^2$ definiert. Dieser Begriff ermöglicht, die obige Ungleichung zu verallgemeinern: Ist $W(w_k = w_k(z_1, z_2), k=1, 2)$ eine pseudokonforme Abbildung des Bereiches $\mathfrak{B}$ in einen Berich $\mathfrak{G} \subset \mathfrak{S}$, so gilt für die B-Inhalte der Flächenlemente von $\mathfrak{L}^2$ und $W(\mathfrak{L}^2)$:

$$(K_{\mathfrak{B}}(z; \bar{z}))^{\frac{1}{2}} \left| \frac{D(z_1, z_2)}{D(u_1, u_2)} \right| du_1 du_2 \geqq (K_{\mathfrak{G}}(w; \bar{w}))^{\frac{1}{2}} \left| \frac{D(w_1, w_2)}{D(u_1, u_2)} \right| du_1 du_2.$$

Die beiden Ungleichungen werden unter gewissen Voraussetzungen noch verschärft. Es wird schliesslich ein dem Spiegelungsprinzip der Klassischen Funktionentheorie nachgebildetes Prinzip der analytischen Fortsetzbarkeit einer Funktion $f(z_1, z_2)$ aufgestellt, mit dessen Hilfe dann noch eine weitere Ungleichung für die nichteuklidischen B-Inhalte von Flächenelementen abgeleitet wird. *P. Thullen.*

Bergman, Stefan. On a generalized Green's function and certain of its applications. Bull. Amer. Math. Soc. **47**, 651–661 (1941). [MF 5061]

Let there be given a domain M of a certain character and possessing a distinguished boundary surface and let there be given a set $\{g_\nu(z_1, z_2)\}$ of analytic functions regular in M and possessing the property A: For every $\rho<1$ there exists an $N(\rho)$ such that $g_\nu(z_1, z_2)$, $\nu>N(\rho)$, does not vanish in $\bar{M}_\rho$, where $\bar{M}_\rho$ is the closure of M_ρ and M_ρ is a certain set interior to M, $M_\rho \to M$ as $\rho \to 1$. The author derives a necessary and sufficient condition for the existence of a function $F(z_1, z_2)$ $(F(0, 0)<\infty)$ belonging to a certain class $l(M)$ and possessing the property that, for each ν, $F(z_1, z_2) + \log |g_\nu(z_1, z_2)|$ is regular on G_ν except at the intersection points $G_\nu \cdot G_n$, $n \neq \nu$. Here G_ν is the set of points in M at which g_ν vanishes. The condition derived is

$$\sum_{\nu=1}^{\infty} \Gamma(0, 0; g_\nu; M) < \infty,$$

where $\Gamma(z_1, z_2; g; M)$ is the Green's function of the extended class corresponding to $g(z_1, z_2)$, a notion introduced by Bergman. The proof uses this notion and that of the extended class of functions which replaces biharmonic functions in this theory. *W. T. Martin.*

Lelong, Pierre. Sur l'intégrale de Kronecker appliquée à un système de deux fonctions de deux variables complexes. C. R. Acad. Sci. Paris **211**, 351–353 (1940). [MF 5353]

The author presents, for the most part without proofs, a series of results on the simultaneous zeros of a pair of analytic functions $f(z, w)$, $g(z, w)$ in a region Δ. The zeros are counted by means of Kronecker's integral taken over the boundary of Δ. The case where f and g are meromorphic in Δ is also considered. For the case in which Δ is the bicylinder and $f \equiv \varphi(z)$, $g \equiv w$, $\varphi(z)$ being meromorphic, the author obtains Shimizu's formula on spherical area in the question of the distribution of zeros of a function of one variable. The reviewer notes that certain of these results are related to some results obtained by D. C. May, Jr. [An integral formula for analytic functions of k variables with some applications," Thesis, Princeton University, 1941; not yet published]. *W. T. Martin* (Cambridge, Mass.).

Lelong, Pierre. Sur les domaines cerclés qui sont domaines naturels d'existence d'une fonction analytique de deux variables complexes. C. R. Acad. Sci. Paris **212**, 426–428 (1941). [MF 4915]

The author presents the statements of some results giving criteria for a Hartogs region to be a region of natural existence of an analytic function. He also concludes some results on the singularities of a function defined by its Hartogs series. *W. T. Martin* (Cambridge, Mass.).

Lelong, Pierre. Sur les zéros d'une fonction entière de deux variables. C. R. Acad. Sci. Paris **211**, 533–535 (1940). [MF 5362]

The author presents some results on the roots $z=b_j(w)$ of $f(z, w)=0$, $f(z, w)$ being analytic. Defining $N(r, w) = \sum_j \log^+ |r/b_j(w)|$, the author shows that $N(r, w)$ can have a definite order of increase in r for almost all values of w, but can have a lower order for a set $E_0(w)$ having the power of the continuum. Other results are given in the bounds of $N(r, w)$ for $|w| \leqq \lambda$ and on lacunary values w_0 of the relation $f(z, w)=0$, that is, values w_0 for which $f(z, w_0)$ is non-vanishing. *W. T. Martin.*

Kreines, M. Sur une classe de fonctions de plusieurs variables. Rec. Math. [Mat. Sbornik] N.S. **9 (51)**, 713–720 (1941). (French. Russian summary) [MF 5505]

Let $\{F\}$ be a family of functions defined in a domain G in real $(x_1, \cdots, x_n)$-space, each F being continuous in G, and having continuous first partial derivatives in G except possibly at a certain class of exceptional sets. Furthermore, let each F be monotone in G, that is, for every domain G_1 which, with its boundary R, is contained in G let the upper and lower bounds of F in $G_1 \cup R_1$ coincide respectively with the upper and lower bounds on R_1. The author proves that the functions of such a family are equicontinuous if there exists a constant $K < \infty$ such that $\int_G \cdots \int |\text{grad } F|^n dv \leqq K$ for every F of the family. The result is applied to quasiconformal representations. *W. T. Martin.*

Oka, Kiyosi. Sur les fonctions analytiques de plusieurs variables. IV. Domaines d'holomorphe et domaines rationellement convexes. Jap. J. Math. **17**, 517–521 (1941). [MF 5644]

[The previous communications appeared in J. Sci. Hirosima Univ. Ser. A. **6**, 245–255 (1936); **7**, 115–130 (1937); **9**, 7–19 (1939).] Im Raume R_{2n} von n komplexen Veränderlichen sind seit langem Beispiele von schlichten Bereichen bekannt, in denen der Rungesche Satz nicht gilt; das heisst, Bereiche, in denen analytische Funktionen existieren, die sich nicht in eine dort gleichmässig konvergente Reihe von rationalen Funktionen entwickeln lassen [vgl. z.B. Cartan-Thullen, Math. Ann. **106**, 617–647 (1932)]. Doch hatten alle diese Bereiche die Eigenschaft, nicht-schlichte Regularitätshüllen zu besitzen; und es blieb bislang die Frage offen, ob der Rungesche Satz nicht wenigstens für schlichte Regularitätsbereiche gelten müsse. Der Verfasser löst dieses interessante Problem, in dem er durch ein konkretes Beispiel die Existenz von schlichten und endlichen Regularitätsbereichen nachweist, in denen der Rungesche Satz nicht gilt. Es ist damit zugleich bewiesen, dass es schlichte und endliche Regularitätsbereiche gibt, die nicht konvex sind inbezug auf rationale, in ihnen analytische Funktionen oder sich nicht vom Innern her durch solche Bereiche approximieren lassen. *P. Thullen.*

Oka, Kiyosi. Sur les fonctions analytiques de plusieurs variables. V. L'intégrale de Cauchy. Jap. J. Math. **17**, 523–531 (1941). [MF 5645]

Falls der Rand eines schlichten und beschränkten Bereiches Δ des Raumes R_4 von zwei komplexen Veränderlichen x, y aus analytischen Hyperflächenstücken $S_1, S_2, \cdots, S_N$ besteht, so lässt sich unter bestimmten Voraussetzungen jede im "Polyeder" Δ analytische Funktion $f(x, y)$ durch ein verallgemeinertes Cauchyintegral ("Weilsches Integral") darstellen:

$$f(x,y) = -\frac{1}{4\pi^2} \sum_{(i,j)} \int_{\sigma_{ij}} \frac{(P_i Q_j - P_j Q_i) f(x,y)}{(X_i - X_i^0)(X_j - X_j^0)} dxdy,$$

wobei das Integral über jede zweidimensionale Schnittfläche σ_{ij} von je zwei Hyperflächenstücken S_i, S_j zu erstrecken ist. Hierbei sind die $X_i(x, y)$, $i = 1, 2, \cdots, N$, unmittelbar durch die Hyperflächen S_i gegebene, in einem Δ ganz umfassenden Bereich D analytische Funktionen; für (x, y) und (x_0, y_0) aus D sind $P_i(x, y; x_0, y_0)$ und $Q_i(x, y; x_0, y_0)$ analytische Funktionen, die der Bedingung:

$$(X_i - X_i^0)R = (x - x_0)P_i + (y - y_0)Q_i$$

genügen, wobei $R(x, y; x_0, y_0)$ eine für (x, y) und (x_0, y_0) aus D analytische Hilfsfunktion ist mit $R(x_0, y_0; x_0, y_0) = 1$. In der vorliegenden Arbeit wird gezeigt, dass jede in einem schlichten endlichen Regularitätsbereiche D analytische Funktion sich im Innern von D durch Weilsche Integrale darstellen lässt; das heisst genauer: D lässt sich vom Innern her durch eine Folge von Polyedern Δ approximieren, von denen jedes Entwicklungen der in D analytischen Funktionen in Weilsche Integrale gestattet. *P. Thullen* (Quito).

Martinelli, Enzo. Sulle funzioni analitiche di più variabili complesse con periodi infinitesimi. Boll. Un. Mat. Ital. (2) **3**, 213–215 (1941). [MF 5590]

Der Verfasser gibt einen neuen Beweis des Satzes von Weierstrass, nach dem eine analytische Funktion von n komplexen Veränderlichen $z_1, z_2, \cdots, z_n$, die eine "infinitesimale Periode" besitzt, stets eine lineare Transformation der z_i zulässt, welche die Zahl der unabhängigen Veränderlichen auf weniger als n vermindert. *P. Thullen* (Quito).

Volovelsky, Sophie. An attempt to construct elements of a theory of functions in one non-commutative hypercomplex system with 3 units. Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **16**, 143–157 (1940). (Russian. English summary) [MF 4742]

This paper deals with a non-commutative hypercomplex system over the real numbers with three units. The author first discusses monogeneity of functions and the generalized Cauchy-Riemann equations. Formulas for differentiation are given, differentiability is defined and power-series expansions are discussed together with questions of their domain of convergence. A hypercomplex integral is defined and a Cauchy theorem on independence of path is proved. The analogue of Cauchy's integral formula is believed not to hold. An extensive literature is cited. *H. Wallman.*

Capelli, Pedro F. Sur le nombre complexe binaire. Bull. Amer. Math. Soc. **47**, 585–595 (1941). [MF 5050]

A binary complex number is defined as a number $a + b\alpha$, with a and b real and $\alpha^2 = \mu + \nu\alpha$. Its distance D from the origin is defined by $D^2 = a^2 + \nu ab - \mu b^2$. The theory of these binary numbers unifies the known special cases of complex numbers ($\alpha^2 = -1$), complex hyperbolic numbers ($\alpha^2 = 1$) and complex dual numbers ($\alpha^2 = 0$). Moreover, every system of binary numbers is reducible to one of these special cases by a suitable choice of oblique axes. The author develops a theory of polygenic functions of binary complex variables patterned on the methods and results of Kasner and De Cicco. *P. Franklin* (Cambridge, Mass.).

GEOMETRY

Wong, Yung-Chow. Some properties of the triangle. Amer. Math. Monthly **48**, 530–535 (1941). [MF 5716]

Krishnaswami Ayyangar, A. A. The theorem of Feuerbach. Math. Student **9**, 16–29 (1941). [MF 5004]

Lebesgue, Henri. Sur une figure projective. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 5–19 (1940). [MF 4398]

The author returns to the consideration of Morley's trisection problem [Enseignement Math. **38**, 39–58 (1940); cf.

these Rev. 2, 8]: "C'est pour mieux examiner le caractère métrique ou projectif de la figure que j'ai repris ici la question en utilisant, tant que je l'ai pu, seulement des raisonnements projectifs. Ce faisant, je n'ajouterai aucun résultat positif à ceux déjà connus; peut-être cependant les réflexions qui suivent intéresseront-elles ceux qui ne se contentent pas de collectionner les faits mathématiques mais aiment à les retourner en tout sens pour les mieux comprendre."
N. A. Court (Norman, Okla.).

Fabricius-Bjerre, Fr. An elementary geometrical problem concerning perspective triangles. Mat. Tidsskr. B. **1941**, 1–17 (1941). (Danish) [MF 5225]
The problem is to find for two given perspective triangles a triangle inscribed in the one and circumscribed to the other. It is shown that the problem has in general two solutions, and that it is equivalent to the problem of finding for a given conic section a polar triangle inscribed in (or circumscribed to) a given triangle. The reality of the solutions is also discussed. *B. Jessen* (Copenhagen).

Sgarbazzini, Carlo. Sui quadrangoli articolati. Period. Mat. (4) **21**, 177–192 (1941). [MF 5387]
The properties of a hinged quadrilateral are presented in an elementary, largely synthetic way. The author does not claim the results to be new and offers references to original sources. *N. A. Court* (Norman, Okla.).

Thébault, V. Quadrangle bordé de triangles isoscèles semblables. Ann. Soc. Sci. Bruxelles. Sér. I. **60**, 64–70 (1940). [MF 5162]

Thébault, V. Octogone bordé de carrés. Mathesis **54**, 114–117 (1940). [MF 5153]

Thébault, V. Polygone de $2n$ cotés bordé de triangles isoscèles semblables. Mathesis **54**, 161–166 (1940). [MF 5159]

Thébault, V. Sur une nouvelle sphère du tétraèdre. C. R. Acad. Sci. Paris **212**, 327–328 (1941). [MF 4905]
The six edges of a tetrahedron $T=ABCD$ give rise to three quadrilaterals $Q=ADCB$, $Q'=ACBD$, $Q''=ABCD$. To these quadrilaterals correspond three tetrahedrons inscribed in T and having their faces respectively perpendicular to the sides of Q, Q', Q''. The author shows that the twelve vertices of these three tetrahedrons lie on a sphere K the center of which is the point the sum of the squares of whose distances from the faces of T is a minimum. He calls K "the second Lemoine sphere." *N. A. Court.*

Bouvaist, R. et Thébault, V. Nouvelles sphères associées au tétraèdre. C. R. Acad. Sci. Paris **211**, 377–378 (1940). [MF 5355]

Delens, Paul. Sur la théorie du tétraèdre. C. R. Acad. Sci. Paris **211**, 220–221 (1940). [MF 5346]
Consider a tetrahedron $\mathfrak{T}=ABCD$ and a hyperbolic group of lines JM_j ($j=A, B, C, D$) passing through the vertices J of $\mathfrak{T}$ and meeting the respectively opposite faces in the points M_j. The lines determine a quadric φ circumscribed about $\mathfrak{T}$ and a quadric P' inscribed in $\mathfrak{T}$, the points M_j being the points of contact. The author states (proofs to be published elsewhere) that by a proper choice of coordinates the two quadrics considered may be shown to belong, respectively, to a point-net and a tangential net of quadrics which remain invariant in a tetrahedral inversion. If $J\bar{M}_j$ is the hyperbolic group, on φ, supplementary to the group JM_j, the lines $M_j\bar{M}_j$ form a hyperbolic group. These groups of lines and points, "solidary" on account of their hyperbolic situation, seem to point to new relations between the geometry of the triangle and the geometry of the tetrahedron. *N. A. Court* (Norman, Okla.).

Delens, Paul. Sur certains éléments permutants du tétraèdre. C. R. Acad. Sci. Paris **211**, 273–275 (1940). [MF 5349]
The author offers supplementary considerations on the geometric elements dealt with in the preceding note, making use of Grassmann's methods. *N. A. Court.*

Brusotti, Luigi. Un semplice modello metrico del tetraesagono. Boll. Un. Mat. Ital. (2) **3**, 310–312 (1941). [MF 5603]

Merz, K. Doppelkreuzhaube. Vierteljschr. Naturforsch. Ges. Zürich **85**, 142–148 (1940). [MF 4947]

Chidambaram, S. On a particular inscribed rectangle of a conic. Math. Student **8**, 143–147 (1940). [MF 4999]

Ciani, Edgardo. Una nuova forma dell'equazione di una conica. Period. Mat. (4) **21**, 107–112 (1941). [MF 5383]

McEwen, W. R. Focal points and focal loci. Amer. Math. Monthly **48**, 386–396 (1941). [MF 5088]

Comét, Stig. Une application des nombres complexes à un problème de la géométrie élémentaire. Proc. [Förhandlingar] Roy. Physiographic Soc. Lund **11**, no. 9, 11 pp. (1941). [MF 5555]
Some relations are found involving the geometric elements of certain simple figures of plane geometry, such as a circle inscribed in a triangle or polygon, or four mutually tangent circles. *P. Franklin* (Cambridge, Mass.).

Venkataraman, B. R. On a new chain of theorems in circle-geometry. J. Indian Math. Soc. (N.S.) **5**, 38–43 (1941). [MF 5112]
Given n circles ($2\leqq n\leqq 7$), whose centers are concyclic. With these circles a point can be associated if n is odd, and a circle if n is even, such that the points or circles associated with each $n-1$ circles contained in the given system are incident with the circle or point associated with all the n circles. If $n=2$, the associated circle is the radical axis of the two circles, and, if $n=3$, the associated point is the radical center. It is not decided whether this theorem holds for $n>7$. *E. Helly* (Long Branch, N. J.).

Freudenthal, Hans. Zur Konstruktion von Tangentenpolygonen. Nieuw Arch. Wiskde (2) **20**, 273–278 (1940). [MF 5211]
This paper deals with the construction of n-agons of tangents to a given polygon with n sides. The author finds conditions on n assuring the construction of the n-agon of tangents by means of ruler and compass. In case $n\leqq 6$ the construction is always possible. The method of proof is a translation of geometric facts into the Galois theory. For n with $n\geqq 7$ an elementary construction in general is impossible. *O. F. G. Schilling* (Chicago, Ill.).

Buzano, Piero. Su una corrispondenza fra curve piane. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 288–295 (1940). [MF 4779]

Given in a projective plane three points A, B, C, functions of a parameter t, not collinear for every value of t. As t varies, each point describes a curve in correspondence for the same value of t. As the points describe curves, the lines joining them in pairs describe envelopes in similar correspondence. The following theorem is proved: When, for a certain value of t, the tangents at A, B, C are concurrent, the points of contact of the lines AB, etc. with their envelopes are collinear. The method of proof is that of the Burali-Forti geometric calculus. The case in which AB, BC, CA always pass through three fixed collinear points is considered in detail. The Cartesian oval belongs to the curves that can be generated in this way [Chasles, Aperçu Historique, 1875, p. 351]. *V. Snyder* (Ithaca, N. Y.).

de Bruijn, N. G. On Steiner-Schläfli's hypocycloid. Nieuw Arch. Wiskde (2) **20**, 282–287 (1940). (Dutch) [MF 5213]

Geometrical consideration concerning a paper by van der Woude [Mathematica, Zutphen B. **8**, 129–134 (1940); these Rev. **2**, 8].

Bagchi, Haridas. On bicircular quartics. J. Indian Math. Soc. (N.S.) **5**, 46–47 (1941). [MF 5114]

"The bicircular quartic is the two-dimensional analogue of the general cyclide, some of whose properties are considered by the present writer in a previous paper" [J. Indian Math. Soc. (N.S.) **4**, 120–124 (1940); these Rev. **2**, 159]. A few theorems concerning polar and focal properties of these curves in general and of Bernoulli's lemniscate and Cassini's oval in particular are stated in this paper, the proofs being omitted as analogous to the corresponding proofs in the earlier paper. *E. Helly* (Long Branch, N. J.).

Srb, Jan. Eine lineare Konstruktion der quadratischen Hyperfläche des n-dimensionalen Raumes aus $n(n+3)/2$ Punkten. Časopis Pěst. Mat. Fys. **70**, 53–67 (1941). (Czech. German extract) [MF 5428]

Richmond, Herbert W. On a chain of theorems due to Homersham Cox. J. London Math. Soc. **16**, 105–107 (1941). [MF 5130]

Attention is called to a chain of theorems dealing with points and planes in space of three dimensions, first given by H. Cox [Quart. J. Math. **25**, 1–77 (1891)] and from which he derived illustrations, in the plane, for Grassmann's Ausdehnungslehre. *N. A. Court* (Norman, Okla.).

Richmond, Herbert W. A chain of theorems for lines in space. J. London Math. Soc. **16**, 108–112 (1941). [MF 5131]

Clifford's chain of points and circles in the plane is transformed by Richmond by means of an inversion whose center lies outside the plane considered. He thus obtains a chain of theorems concerning points and circles on a sphere. By a skillful reinterpretation of the coordinates of the elements involved in this new chain he obtains a chain of theorems for lines in space. *N. A. Court.*

Court, N. A. On the centroid. Nat. Math. Mag. **15**, 271–277 (1941). [MF 5689]

Let $A_1, A_2, \cdots, A_n$ be given points, G their centroid, and G_k the centroid of the $n-1$ points obtained by excluding the point A_k. The author derives a formula for the square of the length of the segment A_kG_k and proves the relations

$$\sum A_kG_k^2 = ns^2/(n-1)^2, \quad \sum A_kG^2 = s^2/n^2, \quad k=1, 2, \cdots, n,$$

where s is the sum of the squares of the $n(n-1)/2$ segments A_iA_j. These formulas are used in order to prove a few known relations concerning spheres through the centroid and three vertices of a tetrahedron. In the third part of the paper theorems are proved involving the distances of the centroid of a tetrahedron from the points of intersection of the faces of the tetrahedron with certain straight lines.

E. Helly (Long Branch, N. J.).

Lense, J. Die sphärische Trigonometrie in der sphärischen Astronomie. Astr. Nachr. **271**, 121–132 (1941). [MF 4611]

The usual formulas of spherical trigonometry are directly deduced using only the transformation formulas of rectangular coordinates. Applications are given by calculating the transformations due to precession, nutation, instrument-error, time equation, etc. *O. Neugebauer.*

Cavallaro, Vincenzo G. Sulla formula che esprime una distanza variabile in funzione di tre distanze assegnate. Boll. Un. Mat. Ital. (2) **2**, 501–502 (1940). [MF 5567]

Weiss, E. A. Metrik in Dreieckskoordinaten. Jber. Deutsch. Math. Verein. **51**, Abt. 2., 9–19 (1941).

The various types of homogeneous coordinates are considered and formulas, in those coordinates, of some metrical elements, like the distance between two points, the area of the basic triangle, etc., are derived in an elegant manner.

N. A. Court (Norman, Okla.).

Watanabe, S. Coordonnées parallèles et leur application I. Jap. J. Math. **17**, 127–138 (1940). [MF 4609]

We take $n+1$ points $A_0, A_1, \cdots, A_n$ in a Euclidean plane π, of which three successive points do not lie in a straight line. On line A_0A_1 a point P_1 is taken, on P_1A_2 a point P_2, on P_2A_3 a point P_3, $\cdots$, which gives an open polygon $P_1P_2\cdots P_n$ which is called the point $p=(P_1P_2\cdots P_n)$. The set of points p is a realization of a projective space of n dimensions $e_{[A]}$ in plane π (to which the points at infinity are added). Another space is obtained by taking $n+1$ parallel lines A_0A_0', A_1A_1', $\cdots$, A_nA_n', of which each passes through the previous points A_i. Take points X_i, $i=0, 1, \cdots, n$, on A_iA_i'; then the points $\{X_i\}$ form a projective space $e_{[X]}$. A space which is formed by all points which are contained in these spaces $e_{[X]}$ is a projective space $E_{[A]}$ of $n+1$ dimensions. If A_nX_n has the length x^n, then the set $(x^0x^1\cdots x^n)$ is called a system of parallel coordinates of $e_{[X]}$. These conceptions form the base of an axiomatic investigation with application to nomography. *D. J. Struik* (Cambridge, Mass.).

Deaux, R. Décompositions d'une homographie binaire. Mathesis **54**, 155–161 (1940). [MF 5158]

Deaux, R. Projectivités ternaires permutables. Mathesis **54**, 97–109 (1940). [MF 5151]

Bottema, O. Self-projective sets of points in R_n. Nieuw Arch. Wiskde (2) **20**, 225–243 (1940). (Dutch) [MF 5208]

Gambier, Bertrand. Étude d'un espace à quatre dimensions décomposable en la somme de deux espaces à deux dimensions. J. Math. Pures Appl. (9) **19**, 237–260 (1940). [MF 4623]

Let p_i $(i=1, 2, \cdots, 6)$ be the homogeneous coordinates of a straight line in 3-space, and $p_1p_4+p_2p_5+p_3p_6=0$ the quadratic relation between them. By a simple linear transformation and norming of the arbitrary factor, new non-homogeneous coordinates P_i can be introduced such that $P_1^2+P_2^2+P_3^2=1$ and $P_4^2+P_5^2+P_6^2=1$. The quantities P_1, P_2, P_3 and P_4, P_5, P_6 can be interpreted as the coordinates of two points a and α, respectively, each on one sheet of the unit-sphere. Two lines (a, α) and (b, β) meet if the spherical distances (a, b) and (α, β) are equal. Since (a, α) and $(-a, -\alpha)$ represent the same line, a mapping of the oriented lines of three-space on the pairs of points (a, α) is obtained. Using this representation, the author gives a new proof of the existence of an isomorphism between the space of the oriented straight lines and the cycles (oriented circles) which are orthogonal to the sphere $x^2+y^2+z^2=1$. This mapping is used to study the cycle-space and its transformations, and to examine certain configurations of cycles and their connection with Dupin's cyclides. *E. Helly.*

Gonseth, F. A propos de la condition d'apolarité de deux complexes de droites. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 225–231 (1940). [MF 4419]

Let $v_1, v_2, \cdots, v_6$ be the six homogeneous coordinates of a straight line in 3-space, $Q(v_1, v_2, \cdots, v_6)=0$ the quadratic relation between them, and $F(v_1, v_2, \cdots, v_6)=0$ the equation of a line complex of order n. This equation can be written in infinitely many ways: $F(v)+M(v)Q(v)=0$, the form $M(v)$ being homogeneous of order $n-2$. In order to obtain a definite equation of the complex, which would permit the definition and study of the apolarity of two complexes, the author suggests determining the function M so that, in the projective space $v_1, v_2, \cdots, v_6$, the manifold $M=0$ is apolar to $Q=0$. He shows, after having proved a few general theorems about apolarity, that M is determined in a unique way, and that the normal form of the equation obtained in this way coincides with a normal form found by Clebsch in an entirely different way. *E. Helly.*

Sintzov, D. Die durch mehrere bilineare Konnexe bestimmten Konfigurationen. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1941**, no. 6, 3–32 (1941). (Ukrainian. Russian and German summaries) [MF 4497]

Study of systems of the bilinear form $\sum a_{ik}x_iu_k=0$ in the ternary and in the quaternary field. In the case of $i, k=1, 2, 3$ the systems considered consist of 1, 2, 3, 4, 5 forms, in the case $i, k=1, 2, 3, 4$ they consist of 1, 2, 3, 4, 5, 6 forms. Three bilinear forms in the ternary field determine a pair of curves of order 3 and class 3, four bilinear forms in this case have 6 elements in common. Four bilinear forms in the quaternary field determine a pair of surfaces of order 4 and class 4, five such form a curve and a developable surface of order 10 and class 10. Six forms have 20 elements in common. *D. J. Struik* (Cambridge, Mass.).

Dedò, Modesto. Su alcune configurazioni projettivamente rigide. Boll. Un. Mat. Ital. (2) **3**, 307–310 (1941). [MF 5602]

Ciani, Edgardo. La configurazione dei cubi iscritti in un dodecaedro regolare convesso. Boll. Un. Mat. Ital. (2) **3**, 313–320 (1941). [MF 5604]

Lyons, R. J. A proof of the theorem of the double-six. Proc. Cambridge Philos. Soc. **37**, 433–434 (1941). [MF 5547]

The author proves the existence of the configuration of Schlaefli's double-six, consisting of two systems of six straight lines each which are coordinated such that each line of either system intersects all the five not corresponding lines of the other system, without the theory of cubic surfaces, using only elementary facts about ranges and reguli. *E. Helly* (Long Branch, N. J.).

White, H. S. Fourteen species of skew hexagons. Bull. Amer. Math. Soc. **47**, 764–768 (1941). [MF 5484]

In each of six planes of general position in real projective space, the traces of the other five bound one convex pentagon (as well as five triangles and five quadrangles). Of the six pentagons in the six planes, each has two sides in common with others. Hence the pentagons, and so also the planes, fall into a natural cyclic order. Let the planes be numbered 1, 2, 3, 4, 5, 6 in one of the twelve ways thus indicated. Any cycle of these numbers (such as 154263) distinguishes a cycle of six lines (namely, 31, 15, 54, 42, 26, 63, where pairs of consecutive planes meet) and a cycle of six points (such as the point 1, where 31 meets 15). Since two points on a line determine two segments, the consequent figure may be regarded in 64 ways as a skew hexagon. If we make the space Euclidean, one such hexagon will have finite edges; thus we may speak of "the hexagon 154263" (which is, of course, the same as 362451 or 315426, etc.). Since the original set of six planes is equivalent to itself under the cyclic permutation (123456), we may transform such a symbol as 154263 by any power of this permutation to derive another skew hexagon of the same "species." In this manner the sixty finite hexagons are distributed into fourteen species, one of which consists of the "normal" hexagon 123456 alone. The paper closes with instructions for determining the species of a given concrete model, and with the suggestion that the theory might be extended to other skew polygons whose vertices lie on a twisted cubic curve. *H. S. M. Coxeter* (Toronto, Ont.).

Walden, Earl. On the mapping of the sets of 24 points of the symmetric substitution group G_{24} in ordinary space upon a hyperquadric cone. Bull. Amer. Math. Soc. **47**, 665–669 (1941). [MF 5063]

A mapping of projective three space with coordinates (x_1, x_2, x_3, x_4) onto a hyperquadric cone Q in projective four-space $(y_1, y_2, y_3, y_4, y_5)$, where $\rho y_1=\sum x_1^4$, $\rho y_2=\sum x_1^2x_2^2$, $\rho y_3=\sum x_1^2x_2^2$, $\rho y_4=\sum x_1^2x_2x_3$, $\rho y_5=x_1x_2x_3x_4$, is obtained, in which the 24 images of a point (x) under the symmetric group G_{24} are mapped on a single point of Q. Eight fundamental F-points in (x) are not mapped in (y), but their infinitesimal neighborhoods are mapped on the vertex line VV' of the hyperquadric cone Q. The four lines $\varphi_1\equiv x_1+x_2+x_3+x_4=0$, $\cdots$, $\varphi_4\equiv x_1x_2x_3x_4=0$, each containing two F points, map into a single point T. These four lines are double tangents to the quartic surface in (x) which is the image of a section of Q by a hyperplane in (y). If the hyperplane contains VV' the quartic factors into two quadrics; if it contains VT, the linear factor $\varphi=0$ splits off; and if it contains $VV'T$, the quartic is the plane $\varphi=0$ counted four times.

The author discusses the images in (x) of the intersection of Q with a hypersurface H_n of degree n, and concludes with a discussion of symmetric quartics. *J. S. Frame.*

Grioli, G. Una proprietà di minimo nella cinematica delle deformazioni finite. Boll. Un. Mat. Ital. (2) **2**, 452–455 (1940). [MF 5564]

Sia C_* un sistema continuo di punti; una trasformazione continua S lo porti in C, un'altra S' in C'. Per un punto generico P_* di C_* scelgo una sfera c_* di centro P_* e raggio ρ cosi piccolo che in c_* gli spostamenti dovuti ad S, S' si possano confondere con i corrispondenti spostamenti omogenei tangenti in P_*. Si chiama divario locale in P_* di S, S' l'integrale

$$\delta(P_*)=\int_{c_*}|QQ'|^2dC_*,$$

ove QQ' è la distanza dei punti Q, Q' corrispondenti in C, C' ad un punto generico Q_* di C_*. L'autore dimostra: Se Δ_1, Δ_2, Δ_3 sono i coefficienti principali di dilatazione lineare di S in P_* e se S' è uno spostamento rigido, allora:

$$\delta(P_*)\geqq 4\pi\rho^5(\Delta_1^2+\Delta_2^2+\Delta_3^2)/15.$$

Il segno $=$ vale soltanto se lo spostamento omogeneo tangente ad S in P_* è il prodotto di una pura deformazione per S'. Il fattore $4\pi\rho^5/15$ è il valore di $\int_{c_*}y^2dC_*$, ove y è una delle coordinate cartesiane di un punto di c_*, quando P_* sia l'origine. Il calcolo, molto semplice, è scritto in forma simbolica, ma si potrebbe facilmente esplicitare con altre notazioni. *G. Fubini* (Princeton, N. J.).

Chern, Shiing-shen e Yien, Chih-ta. Sulla formula principale cinematica dello spazio ad n dimensioni. Boll. Un. Mat. Ital. (2) **2**, 434–437 (1940). [MF 5561]

This paper extends to Euclidean n-space Blaschke's "principal kinematic formula" [Atti Accad. Naz. Lincei. Rend. (6) **23**, 546–547 (1936)]. Let R_0, R_1 be a fixed and movable n-domain, $K(R_0\cdot R_1)$ the total curvature of the frontiers of their intersection, $\dot{R}_1$ the kinematic density of Poincaré. The generalization expresses $\int K(R_0\cdot R_1)\dot{R}_1$ extended to all positions of R_1 as a certain bilinear form in M_i^k and V_i. Here M_i^k $(k=0, 1, \cdots, m-1)$ is the integral over the frontier of R_i of the kth symmetric function of the principal curvatures; V_i is the volume of R_i.

J. L. Vanderslice (Bethlehem, Pa.).

Wu, Ta-Jen. Der Dual der Grundformel in Integralgeometrie. J. Chinese Math. Soc. **2**, 199–204 (1940). [MF 4206]

If a surface $\mathfrak{F}$ is given, a tangential element $\mathfrak{T}=\{\mathfrak{E};\mathfrak{e}\}$ consists of the tangent plane $\mathfrak{E}$ in a point $\mathfrak{z}$ of $\mathfrak{F}$ and a tangent vector $\mathfrak{e}$ of length 1 in $\mathfrak{z}$. We may define the density $\dot{\mathfrak{T}}$ of $\mathfrak{T}$ with respect to $\mathfrak{F}$ as the alternating product $\dot{\mathfrak{T}}=\dot{\lambda}_1\dot{\lambda}_2\dot{\lambda}_3$, where $\dot{\lambda}_1$, $\dot{\lambda}_2$, $\dot{\lambda}_3$ are respectively the infinitesimal rotations of $\mathfrak{E}$ around $\mathfrak{e}$, a tangent in $\mathfrak{z}$ orthogonal to $\mathfrak{e}$, and the normal of $\mathfrak{F}$ in $\mathfrak{z}$. Let the surface $\mathfrak{F}$ be fixed, the surface $\mathfrak{F}'$ be movable. We consider the tangential elements $\mathfrak{T}=\{\mathfrak{E};\mathfrak{e}\}$, $\mathfrak{T}'=\{\mathfrak{E}';\mathfrak{e}'\}$, where $\mathfrak{E}=\mathfrak{E}'$ is the tangent plane of $\mathfrak{F}$ in $\mathfrak{z}$ and of $\mathfrak{F}'$ in $\mathfrak{z}'$ and $\mathfrak{e}$ and $\mathfrak{e}'$ lie in the straight line through $\mathfrak{z}$ and $\mathfrak{z}'$ and have the same direction. Let t be the infinitesimal rotation of $\mathfrak{E}=\mathfrak{E}'$ around this line and $\dot{\mathfrak{F}}'$ be the kinematic density of $\mathfrak{F}'$ [Blaschke, Integralgeometrie II, 1937, p. 64]. Then

$$t\dot{\mathfrak{F}}'=\mathfrak{p}\dot{\mathfrak{T}}\dot{\mathfrak{T}}'i, \qquad t=|\mathfrak{z}-\mathfrak{z}'|.$$

This formula may be considered as a dual analogue of Blaschke's "Kinematische Grundformel" [ibid. p. 122]. If $\mathfrak{F}'$ degenerates into a point $\mathfrak{p}$, it can be simplified to

$$t\dot{\mathfrak{p}}=\mathfrak{p}\dot{\mathfrak{T}}i,$$

where $\dot{\mathfrak{p}}$ is the density of the point $\mathfrak{p}$.

From the latter formula various integral formulas are derived for convex and continuously curved $\mathfrak{F}$ and $\mathfrak{p}$ belonging to a bounded domain that satisfies different conditions in the various cases. *P. Scherk.*

Convexities, Extremal Problems

Blaschke, Wilhelm. Questioni sui corpi convessi. Boll. Un. Mat. Ital. (2) **3**, 223–230 (1941). [MF 5593]

Burckhardt, Joh. Jak. Über konvexe Körper mit Mittelpunkt. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 149–154 (1940). [MF 4413]

Simple proofs are given for the following two theorems due to Minkowski and A. D. Alexandroff, respectively: (1) A convex polyhedron in n-space, which can be composed of polyhedra which have a center, has itself a center. (2) A convex polyhedron in three space whose faces are (convex) polygons which have a center has itself a center.

H. Busemann (Chicago, Ill.).

Robinson, C. V. A characterization of the disc. Bull. Amer. Math. Soc. **47**, 818–819 (1941). [MF 5495]

If the disc (circle plus its interior) will cover each three points of a given subset of the plane, it will cover this subset. The disc is the only connected, simply connected domain (closure of a bounded open subset) of the plane which has this property. *J. L. Dorroh.*

Scherrer, W. Eine Kennzeichnung der Kugel. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 40–46 (1940). [MF 4402]

The author proves: The total torsion $T=\oint\tau(s)ds$ ($s=$arc length, $\tau(s)=$torsion) vanishes for all the closed curves on a surface S if and only if S is a sphere (or plane) or part of it. The proof of the second part of this theorem is based upon the condition for the variation of T on S to vanish. [From the proof it follows that (1) it is sufficient to suppose T independent of the choice of the curve; (2) it is sufficient to suppose that the variation of T vanishes along all the plane cuts of S.] *P. Scherk.*

Auerbach, H. Sur une propriété caractéristique de l'ellipsoïde. Studia Math. **9**, 17–22 (1940). (French. Ukrainian summary) [MF 5253]

The author states the theorem: Let S be a bounded closed surface in the $(n+1)$ dimensional $(n\geqq 1)$ affine space and suppose that for any couple of points P, P' of S there exists a projectivity which maps S onto itself and carries P into P'; then S is an ellipsoid. It is first shown that the said projectivities form a Lie group with an invariant connected subgroup G; then it is shown that S is analytic and connected. The projectivities are then represented by linear forms in $n+2$ homogeneous coordinates with determinant 1. Using an invariant J, deduced from Fubini's differential forms, the author proves that the coefficients

of the forms are bounded. The proof is accomplished by the construction of a positive quadratic form which is invariant under G. [It seems to the reviewer that this interesting argument should be supplemented (on p. 20) by a consideration of surfaces other than ellipsoids which contain exceptional points in which Fubini's form F_3 vanishes identically.] *G. Fubini* (Princeton, N. J.).

Krein, M. and Milman, D. On extreme points of regular convex sets. Studia Math. **9**, 133–138 (1940). (English. Ukrainian summary) [MF 5262]

The principal result of this paper is the following theorem: Let $K \subset E$ be a bounded regularly convex set [for "regularly convex," see these Rev. **1**, 335]. Then the set of extreme points of K (that is, the points which are not inner points of segments of K) is not empty and its regularly convex envelope coincides with K. A consequence is: If the unit sphere of an infinite-dimensional space E has only a finite number of extreme points, then E is not adjoint to any Banach space. Two applications of this are given. *J. V. Wehausen* (Columbia, Mo.).

Dinghas, Alexander. Über eine Verschärfung der isoperimetrischen Ungleichung in der Ebene. Akad. Wiss. Wien, S.-B. IIa. **149**, 117–132 (1940). [MF 4937]

In Hurwitz' classical proof, the isoperimetric inequality for plane areas is reduced to the lemma: $\int_0^{2\pi}(y'^2-y^2)d\varphi \geqq 0$ for a function $y(\varphi)$ of period 2π with $\int_0^{2\pi} y d\varphi = 0$. The author generalizes the isoperimetric inequality by proving the following more general lemma: For an n-times continuously differentiable function $y(\varphi)$ of period 2π with $\int_0^{2\pi} y d\varphi = 0$, we have

$$0 \leqq \sum_0^{n-1} k_\lambda A_{n-\lambda-1} \leqq \frac{1}{(n+1)^2} \sum_0^{n-1} k_\lambda A_{n-\lambda},$$

if

$$A_\lambda = \tfrac{1}{2}\int_0^{2\pi} [(y^{(\lambda+1)})^2 - (y^{(\lambda)})^2] d\varphi, \quad k_\mu = \sum_0^{\mu} (-1)^\lambda S_\lambda,$$

where S_λ denotes the λth elementary symmetric function of the numbers $1^2, 2^2, 3^2, \cdots, n^2$ (proof from completeness of trigonometric polynomials). If this lemma is applied to the function of support $y = p(\varphi)$ of a convex curve C, A_0 becomes the isoperimetric defect of C, $|A_1|$ the area of the evolute C_1 of C, $|A_2|$ the area of the evolute C_2 of C_1, etc. The author also generalizes his lemma to functions on hyperspheres, using completeness of spherical harmonics. *F. John* (Lexington, Ky.).

Blumenthal, L. M. and Wahlin, G. E. On the spherical surface of smallest radius enclosing a bounded subset of n-dimensional euclidean space. Bull. Amer. Math. Soc. **47**, 771–777 (1941). [MF 5486]

The authors prove (1) if M is any bounded subset of n-dimensional Euclidean space E_n (containing more than a single point) there exists a unique $S_{n-1,r}$ (that is, $(n-1)$-dimensional spherical surface of an n-dimensional sphere of radius r in E_n) of smallest radius r enclosing M and (2) if d is the diameter of M, then the radius of the unique smallest $S_{n-1,r}$ enclosing M satisfies the relation $r \leqq [n/2(n+1)]^{\frac{1}{2}} \cdot d$. These results were established by H. W. E. Jung [J. Reine Angew. Math. **123**, 241–257 (1901)] by analytical methods of considerable complexity. In the present treatment the first step is the proof, by means of a theorem of Helly on convex bodies in E_n, of the lemma: if each set of $n+1$ points of a subset M of E_n is enclosable by $S_{n-1,r}$ of given radius r, then M itself is enclosable by this $S_{n-1,r}$. This lemma permits the reduction of the problem to a finite one concerning $n+1$ points. Two further lemmas establish essentially (1) and (2) when M consists of $n+1$ points in E_n; their proofs are elementary. Then induction and the lemmas are used to establish (1) and (2) in the general case. The paper closes with an account of the history of the problem, its proofs, variations, extensions and applications. *G. B. Price* (Lawrence, Kans.).

Fejes, Ladislas. Sur un théorème concernant l'approximation des courbes par des suites de polygones. Ann. Scuola Norm. Super. Pisa (2) **9**, 143–145 (1940). [MF 5458]

Let K be a closed convex plane curve and P a polygon in the same plane. Each of these lines divides the plane in an interior and an exterior part. Let D be the set of all points belonging to the interior of one and only one of the two lines. The author calls the area of this set the "deviation" of P and K, and proves the following theorem: It is possible to construct a series of polygons $P_3, P_4, \cdots, P_n, \cdots$ of $3, 4, \cdots, n, \cdots$ sides such that

$$\lim n^2 \tau_n \leqq (\pi^2/4)T, \qquad n \to \infty,$$

τ_n being the deviation of K and P_n, and T the area bounded by K. The sign of equality holds only if K is an ellipse. A similar statement can be made concerning the length of K. *E. Helly* (Long Branch, N. J.).

Freudenthal, Hans. Überdeckungen des Einheitskreises mit untereinander kongruenten Mengen. Nieuw Arch. Wiskde (2) **20**, 279–281 (1940). [MF 5212]

Let C be the unit circle, and for $A \subset C$ let A_ϕ denote A turned by the angle ϕ. (1) Given $L < 1$ and any integer p, there is a set A composed of a finite set of intervals, of total length L, such that, for any angles $\phi_1, \cdots, \phi_p$, $A_{\phi_1} \cup \cdots \cup A_{\phi_p} \neq C$. (2) Let $\nu(A) = \inf_\phi \mu(A \cap A_\phi)$ (μ = measure). Then $2\pi\nu(A) \leqq (\mu(A))^2$. *H. Whitney.*

Hadwiger, H. Überdeckung ebener Bereiche durch Kreise und Quadrate. Comment. Math. Helv. **13**, 195–200 (1941).

The author proves very elegantly that any plane region of circumference L and area F can be covered by $[(2/\pi)L + F + 1]$ unit squares (that is, squares of unit side), or by $[(2/\pi\sqrt{3})L + (2/3\sqrt{3})F + 1]$ unit circles (that is, circles of unit radius). He also obtains the sufficient conditions $L - \pi F > (L^2 - \pi^2 F)^{\frac{1}{2}}$, $L - 2F > (L^2 - 4\pi F)^{\frac{1}{2}}$ for a plane region to be covered by a single unit square or unit circle, respectively, and shows by an example that no such condition (involving L and F alone) could be necessary as well as sufficient. *H. S. M. Coxeter* (Toronto, Ont.).

Santaló, L. A. The mean value of the number of parts into which a convex domain is divided by n arbitrary straight lines. Revista Union Mat. Argentina **7**, 33–37 (1941) = Union Mat. Argentina, Publ. no. **17**, 7 pp. (1941). (Spanish) [MF 4455]

Cutting an oval K with n variable straight lines $G_1, \cdots, G_n$, the author determines the mean values $\bar{N}$, $\bar{N}'$, $\bar{\lambda}$, respectively, of the numbers N of domains into which the G_i divide K, N' of points of intersection of the G_i that fall into K, λ of (straight or curved) sides of each of the domains. Let F be the surface of K, L the length of its circumference. Then

$$\bar{N}' = \int N' dG_1 \cdots dG_n / \int dG_1 \cdots dG_n$$

by definition; here dG_i means the measure of the straight line G_i, and the integrals are extended over all the positions for which $K \cdot G_i \neq 0$ $(i=1, \cdots, n)$. Writing $N'=\sum N'_{ij}$ ($N'_{ij}=1$ for $G_iG_jK \neq 0$, $N'_{ij}=0$ elsewhere), the author readily concludes from $\int N'_{ij}dG_idG_j=2\pi F$ that $\bar{N}'=n(n-1)\pi FL^{-2}$. The numbers N and N' being interconnected by the formula $N=N'+n+1$, he finds $\bar{N}=\bar{N}'+n+1$, finally $\bar{\lambda}=4(1-1/\bar{N})$.

P. Scherk (Bloomington, Ind.).

MECHANICS

Astronomy

Kopal, Zdeněk. The form of gaseous configurations rotating with non-uniform angular velocity. Proc. Nat. Acad. Sci. U.S.A. **27**, 359–365 (1941). [MF 5072]

The author shows that in a slowly rotating spheroid with uniform angular velocity ω and mean equatorial radius a the oblateness $e=1-b/a$ lies between $3v/4$ and $15v/4$, where $v=\omega^2/(2\pi G\rho)$, G being the gravitational constant and ρ the mean density. In case ω is not uniform, the lower bound remains the same but the upper bound may increase. These results are applied to the discussion of the external form of slowly rotating stars. *B. Friedman* (Chicago, Ill.).

Severny, A. B. On the stability of Emden's compressible heterogeneous spheres. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 405–408 (1941). [MF 4379]

The author considers the stability of hydrostatic equilibrium for a sphere of continuous compressible matter subject to its own gravitation. The sphere is composed of a perfect gas with constant internal specific heat (polytropic Emden sphere), assuming that the density is a maximum at the center of the sphere and that it is finite at that point. Only radial perturbations are considered in this note. When the pressure-density relation has the form $p=K\rho^\gamma$, and when n is the "index of the polytrope," the conclusion is reached (1) that the star is stable if $\gamma>4/3$ (Eddington's model); (2) Emden's spheres with $n<3$ are stable; (3) Emden's spheres with $n=3$ are in neutral equilibrium; (4) Emden's spheres with $n>3$ are unstable. *E. J. Moulton.*

Severny, A. B. On the stability of Emden's compressible heterogeneous spheres. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 106–108 (1941). [MF 4695]

The note discusses the relative stability of Emden's spheres while in hydrostatic equilibrium, under arbitrary perturbations. Let n be the polytrope index, and let l be the order of a spherical harmonic perturbation; the author's conclusions may be stated thus: (1) any Emden sphere with $n<3$ is stable with regard to any vibration; (2) sphere $n=3$ is in neutral equilibrium with regard to purely radial vibrations ($l=0$) and stable with regard to any vibration $l\geqq 1$; and (3) any Emden sphere with $n>3$ is unstable.

E. J. Moulton (Evanston, Ill.).

Severny, A. B. On the stability of gaseous stars and white dwarfs. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 109–110 (1941). [MF 4696]

The note is concerned with the stability of a non-rotating spherical star when in hydrostatic equilibrium, leaving out of consideration the viscosity of the star substance and also irreversible heat generation. In the case of white dwarf stars, with non-relativistic degeneracy of gas, the author concludes that the problem of stability coincides with that of a polytropic Emden sphere $n=\frac{3}{2}$ [see the preceding review]; and hence a white dwarf is in neutral equilibrium.

E. J. Moulton (Evanston, Ill.).

Mattioli, G. D. Sopra gli effetti secolari delle maree e dell'irraggiamento di massa sugli elementi del moto dei sistemi binari e il problema dell'evoluzione secolare delle stelle doppie. Pont. Acad. Sci. Comment. **4**, 225–286 (1940). [MF 4111]

The author takes up the study of the secular effect of tides on the motion of celestial bodies, particularly as regards the theory of double stars, where the masses and physical characteristics of both components are similar and Darwin's tidal theory does not apply. He attempts then to find out if a slow tidal action superposed on a slow loss of mass due to the radiation of energy from both stars can lead to a plausible view of the evolution of a double star. Recent hydrodynamic research leads him to assume that tidal dissipation is turbulent, so that the coefficient of turbulent transport can be estimated from astrophysical data. The investigation of the duration of evolution is carried through from the standpoint of Levi-Civita's application of adiabatic invariants [Atti Cong. Intern. Mat. Bologna, 1928, vol. 5, p. 17] to astrophysical problems and of the stationary solutions due to Krall [Atti Accad. Naz. Lincei. Rend. **15**, 219, 371, 664 (1932)]. There is an interesting discussion of the conditions of stability and instability. The theory leads to results in agreement with observation for spectroscopic binaries of class B for which the duration of evolution is calculated to be 10^9 years and the initial period about 5 days, but leads to difficulties for class A stars of large period and for binaries of classes G and K for which it would be expected that such stars of period around 100 days should be very numerous and of low eccentricity.

M. S. Vallarta (Cambridge, Mass.).

Camm, G. L. The ellipsoidal distribution of stellar velocities. Monthly Not. Roy. Astr. Soc. **101**, 195–215 (1941). [MF 5115]

The work is concerned with the problem of stellar dynamics in the form treated last and most extensively by Chandrasekhar. The velocity-distribution function is assumed to be a function of a quadratic form in the residual velocities. Steady systems, with non-vanishing mean velocities, are considered. The author shows that the equation of continuity demands a rotationally symmetrical gravitational potential. The more general helical symmetry included in Chandrasekhar's work is impossible in a finite system. It is shown that the vertex of the star streaming cannot deviate in longitude from the direction to the center. The general solution for the potential in the rotationally symmetrical system is given.

The author next attacks the problem which has been neglected in earlier work, namely of checking whether the obtained gravitational potential and distribution function are comparable with Poisson's equation. Assuming no external gravitating effect, and the mass-density to be proportional to star-density, Poisson's equation introduces a further relation between the potential and distribution function obtained from the equation of continuity. It is shown that under the given conditions it is not possible to

reconcile any possible steady solution of the equation of continuity with Poisson's equation. *G. Randers.*

Hydrodynamics, Aerodynamics, Acoustics

Hopf, Eberhard. Ein allgemeiner Endlichkeitssatz der Hydrodynamik. Math. Ann. 117, 764–775 (1941). [MF 5656]

A viscous incompressible fluid is considered which fills a region G bounded by a finite number of surfaces each of which moves as a rigid body. For the resulting flow certain inequalities are derived, which imply limitations on the kinetic energy. Let u_1, u_2, u_3 be the components of the velocity, which as functions of x_1, x_2, x_3 satisfy the Navier-Stokes differential equations. Let

$$2K(u)=\iiint_G \sum_i u_i^2 dx_1 dx_2 dx_3,$$

$$I(u)=\int_t^{t+1} dt \iiint_G \sum_{i,\nu} (\partial u_i/\partial x_\nu)^2 dx_1 dx_2 dx_3.$$

Assume that the velocities of the boundaries and the reciprocal distances between them remain bounded. Then the statement is that $K(u)$ and $I(u)$ remain bounded. For the proof certain ideas are extracted from a paper by Leray [J. Math. Pures Appl. (9) 12, 1–82 (1933)], where a similar problem for stationary boundaries is treated. Functions h_1, h_2, h_3 of x_1, x_2, x_3 are constructed which coincide at the boundaries with u_1, u_2, u_3 and which among others permit the estimate $|\partial h_i/\partial x_\nu| \leqq \epsilon s^{-2}$ near the boundary, $s(x)$ being the distance of the point x_1, x_2, x_3 from the boundary and ϵ being arbitrarily small. Then simple integral inequalities for $K(u-h)$ and $I(u-h)$ yield the statement.

K. Friedrichs (New York, N. Y.).

Lichnerowicz, André. Sur un théorème d'hydrodynamique relativiste. C. R. Acad. Sci. Paris 211, 117–119 (1940). [MF 5339]

This paper deals with the motion of a perfect fluid in static space-time with metric $Udt^2+g_{ij}dx^i dx^j$, $i, j=1, 2, 3$, the coefficients being independent of t. Assuming the existence of a steady motion with p and ρ independent of t, the author establishes a generalization of the classical theorem of Bernoulli in the form $UF^2(1+u^2)=\text{const.}$ along a stream line. Here u is the magnitude of the spatial velocity vector and F is the index $F=\exp \int_{p_0}^{p} dp/\rho$. When p/ρ and the velocity of the fluid are small, this gives approximately

$$\tfrac{1}{2}U+(\tfrac{1}{2}u^2+\textstyle\int_{p_0}^{p} dp/\rho)U=\text{const.}$$

This extends to static space-time a result established by Synge for flat space-time. The method, which is very simple and direct, is based on a theorem due to Eisenhart.

J. L. Synge (Toronto, Ont.).

v. Koppenfels, Werner. Two-dimensional potential flow past a smooth wall with partly constant curvature. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 996, 19 pp. (1941). (10 figures) [MF 5766]

English translation of a paper in Luftfahrtforschung 17, 189–195 (1940); cf. these Rev. 2, 168.

Hamel, Georg. Über die Potentialströmungen zäher Flüssigkeiten. Abhandlungen zur Hydrodynamik. IX. Z. Angew. Math. Mech. 21, 129–139 (1941).

Solutions of the Stokes-Navier equations corresponding to potential flow are generally believed to have little physical bearing since, assuming curl $\bar{v}=0$: (1) the Stokes-Navier equations integrate to the Bernoulli equation and thus the energy exchange appears to be the same as in the case of the frictionless fluid; (2) the non-slip condition on solid, stationary boundaries leads only to the trivial solution $\bar{v}=0$ throughout the fluid. In order to disprove this general belief the author discusses first the integral expression for the dissipation L_i and for the work done per unit time by the frictional forces on the boundary L_a:

$$L_i=-\iiint\left[X_x\frac{\partial v_x}{\partial x}+X_y\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+X_z\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)+Y_y\frac{\partial v_y}{\partial y}+\cdots\right]dV,$$

$$L_a=\oint (X_n v_x+Y_n v_y+Z_n v_z)dF.$$

In the case of potential flow $L_i=L_a$, which is a result corresponding to the existence of the Bernoulli equation (1). In two dimensional flow the expressions for L_i and L_a can be simplified by introducing a complex stream function $f(z)=\varphi+i\psi$. Thus

$$L_i=-4\mu\iint f'\overline{f'}\,dxdy, \quad L_a=i\mu\oint (f'd\overline{f'}-\overline{f}df').$$

Several applications of these formula are given; the rotating cylinder, the flow around one and two stationary cylinders and around a Joukowsky airfoil are discussed. In order to satisfy the non-slip condition (2) the solid bodies have to be assumed to be bounded by a flexible, moving "skin." Only the case where the angular velocity is constant along the boundary can be satisfied by a rigid boundary (for example, the rotating circular cylinder). This leads to the problem of finding cylindrical profiles such that the angular velocity on the boundary is constant. This question is discussed in part II of the paper; it leads to the following non-linear boundary problem: to find a holomorphic function $w=f(z)$ for $w>\rho$ which satisfies the condition

$$\left|\frac{dz}{dw}\right|^2=\frac{C}{2\pi\rho^2\omega}\Re\left\{\frac{w}{z}\frac{dz}{dw}\right\}$$

at $w=\rho$, where ω is the angular velocity. Only the solution for the circular cylinder $wC_0=z$, which has been discussed in part I, can be given at the present time. It is furthermore shown that there exist no "neighboring" solutions of the form

$$z=C_0\sum_{n=0}^{\infty}\epsilon^n g_n(w),$$

where $g_0(w)=w$ and ϵ is sufficiently small.

H. W. Liepmann (Pasadena, Calif.).

Simonov, L. A. Construction of shapes by means of hodographs of velocities. J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] (N.S.) 4, no. 4, 97–116 (1940). (Russian. English summary) [MF 4648]

The construction of the profile corresponding to a given hodograph of velocities is discussed in the paper. The author considers flows with and without circulation for a single profile and for a lattice. He indicates a procedure for the construction of hodographs, corresponding to different angles of attack from the hodograph for a particular value

of this angle; the exterior of profiles of a lattice is mapped onto (infinitely many times covered) upper and lower half planes; the boundaries of profiles go into the real axes [see König, Z. Angew. Math. Mech. 2, 422 (1922)]. In the latter case the changes of the velocity for different angles of attack can be easily determined. The above mentioned procedure consists in transferring the velocity distribution obtained to the plane of profiles. Several examples are discussed. *S. Bergman* (Providence, R. I.).

Serini, Rocco. Deduzione delle equazioni della dinamica dei sistemi continui senza far uso del principio di D'Alembert. Boll. Un. Mat. Ital. (2) 3, 281–283 (1941). [MF 5597]

In the mathematical statements of the principles of linear and angular momentum for a continuous system, the operator d/dt is applied to volume integrals. The paper deals with the process of bringing the operator under the sign of integration. It appears to the reviewer that the results are immediate, in view of the invariance of the mass of an elementary volume moving with the medium.

J. L. Synge (Toronto, Ont.).

Serini, Rocco. Risultante e momento risultante delle azioni capillari su un pezzo di superficie. Boll. Un. Mat. Ital. (2) 3, 207–210 (1941). [MF 5588]

Agostinelli, Cataldo. Sul moto per sola gravitazione di un mezzo continuo disgregato. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 76, 3–20 (1941). [MF 4791]

The author considers motion of particles in a medium statistically equivalent to a continuous medium. It is assumed that the particles are subject only to gravitational forces with no inter-molecular forces acting. The equations of motion of an individual particle are obtained and solved on the assumption that the density of the medium and the velocity components of the particles vary either in concentric spherical layers or in coaxial circular cylindrical ones. It is shown that there are limited regions in which the motion exists, and that it is not possible for the particles actually to attain the center of the spherical system or the axis of the cylindrical one. *M. C. Gray.*

Agostinelli, Cataldo. Vortice cilindrico in un fluido viscoso. Boll. Un. Mat. Ital. (2) 3, 283–286 (1941). [MF 5598]

In a previous paper [Atti Accad. Naz. Lincei. Rend. (6) 23, 670–676 (1936)] the author derived a solution of the Eulerian equations for an incompressible fluid of the form

$$(1)\quad \bar{v}=aq_1^{-1}\operatorname{grad} q_1+[\omega_0 q_1^2+f(t)]\operatorname{grad} q_2 +[F(t)-bq_1^2]\operatorname{grad} q_3,$$

where q_1, q_2, q_3 are cylindrical coordinates; a, ω_0, b are arbitrary constants and $f(t)$, $F(t)$ are functions of the time. The velocity field (1) represents a vortex with heliocoidal vortex lines given by (2) $q_1^2=\text{const.}$, $q_3=\omega_0 q_2/b+\text{const.}$ The vortex is supposed to be bounded by the cylindrical surface (3) $q_1^2-2at=\text{const.}$ In the present paper the author proposes to show that in a viscous fluid the heliocoidal vortex (1) degenerates into a cylindrical vortex with straight vortex lines parallel to the axis q_3. Substituting (1) into the Stokes-Navier equations, the term accounting for the viscous forces becomes

$$(4)\qquad \nu \operatorname{curl}\operatorname{curl}\bar{v}=\operatorname{grad}[4b\nu q_3].$$

The author computes the pressure inside and outside of the vortex and concludes that in order to obtain an uniform pressure distribution throughout the field the constant b must vanish. Hence according to (1) and (2) the vortex reduces to a cylindrical vortex. (4) shows that in this case the action of the viscous forces disappears.

H. W. Liepmann (Pasadena, Calif.).

Ginsburg, I. P. On the theory of ship-waves and wave-resistance. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 7, 129–160 (1939). (Russian) [MF 3336]

The author considers a ship moving with a constant velocity v_0 on the surface of an incompressible fluid. Under the usual hypotheses, the study of waves which arise in such a motion can be reduced to the determination of an harmonic function $\phi(x, y, z)$ satisfying the following boundary conditions: $\phi_{zz}=-g\phi_x/v_0^2$ for $z=0$, $\partial\phi/\partial n=v_n$ on the surface S of the ship, and $\phi\to 0$ for $r\to\infty$; v_n denotes the normal component of v_0. The author gives a critical survey of the solutions by Michell [Philos. Mag. 145 (1898)] and Havelock [Proc. Roy. Soc. London. Ser. A. 138, 339–348 (1932)]. The expression of Havelock (a certain Fourier integral) is the solution of the equations of hydrodynamics given by Lamb and cannot be used for a perfect fluid. Starting from a representation of ϕ as the real part of a Fourier integral the author obtains for ϕ

$$\phi=\int\int \sigma(x_1, y_1, z_1)\left\{\partial\left[\sum_{k=1}^{2}(-1)^{k+1}(x-x_1)(y-y_1+r_k)^{-1}\right]/\partial y +g\left[\sum_{k=1}^{2}(-1)^{k+1}(z+(-1)^k z_1)(y-y_1+r_k)^{-1}\right]/v_0^2\right\}dS,$$

where $x_1, y_1, z_1=\phi(x_1, y_1)$ are the coordinates of the surface S of the ship, and $r_k^2=[(x-x_1)^2+(y-y_1)^2+(z+(-1)^k z_1)^2]$; $\sigma(x_1, y_1, z_1)$ is the solution of an integral equation of Fredholm type of second kind. A proof of the existence of σ is given. The solution ϕ is discussed, the formula of wave-resistance is given, some physical conclusions are drawn and the results are compared with those of Joukowski, Hogner and Havelock. The case of divergent waves is considered.

S. Bergman (Providence, R. I.).

Minorsky, Nicholas. Control problems. J. Franklin Inst. 232, 451–487, 519–551 (1941). [MF 5710]

This paper is essentially a monograph on the treatment of control problems. The dynamic implication of control problems is carefully defined and the mathematical machinery necessary to treat such problems is discussed. An idea of the contents of this monograph may be had from its table of contents. Part I. Application of Operators to Linear Control Problems. Elements of controlled system; degrees of freedom. Differential control operator. Distributive rule of operators; resultant operator. Integrated control terms in operational notations. Direct and indirect control operators; non commutativity of operators. Symbolic multiplication of operators by δ^λ. Differential equation of controlled process; effect of integrated terms in the operator. Examples of application of operators to control problems. Part II. General Properties of Linear Systems Acted upon by Linear Controls. Classification of control systems. Fundamental control system and its properties. Non-fundamental control system; effect of residue operators on the controlled process. Principle of superposition; control terms. Control matrix and its reduction to direct control terms. Sheared matrix of control terms; equivalent control terms. Examples of equivalent control terms; floating zero in a

controlled process. Examples of application of operators to the analysis of various control terms. Controlled pendulum. Part III. Stability of Controlled Systems. General considerations. Inherent stability, "controlled stability" and stability of control system per se. . . . Hurwitz's theorem and its applications. Retarded control action; hystero-differential equation of a retarded control process. Stability of controlled systems; asymptotic instability; self-excited oscillations. Anticipatory control, effect of over-compensation. Extension of operational notations to retarded controls. Parasitic frequencies spectrum of a retarded control process; graphical solution of transcendental characteristic equation. Part IV. Non-Linear Control Problems. Functional control operators. Examples of a non-linear control; method of integrating contour and its applications to functional transformations. Stability of non-linear control systems; Liapounoff's theorem; method of the first approximation. Effect of a retarded action in non-linear control systems. Non-linear systems acted upon by non-linear controls. Applications of non-linear control systems. A bibliography is appended. *A. E. Heins* (West Lafayette, Ind.).

*Bakhmeteff, Boris A. **Coriolis and the energy principle in hydraulics.** Theodore von Kármán Anniversary Volume, pp. 59–65. California Institute of Technology, Pasadena, Calif., 1941.

The paper deals mainly with the controversial coefficient of correction with which the "average velocity head" $U^2/2g$ is to be multiplied in order to arrive at a quantity representative of kinetic energy conditions of a cross-section as a whole. After a review of the historical development of the problem, from the pioneer work of Bernoulli, through Coriolis' investigations to the publications of Saint Venant and of Boussinesq, the author attempts a new conceptual classification and comes to the conclusion that "whenever the problem is to compare energy-conditions at different stations" the average velocity head $U^2/2g$ should be multiplied by the Boussinesq coefficient $\int u^3 dF/U^3F$ rather than, as is usually done, by the Coriolis coefficient $\int u^3 dF/U^3F$; this latter coefficient is to be used only if the problem calls for "appraising the energy flux over an interval of time."

P. Nemenyi (Fort Collins, Colo.).

Sears, William R. Some aspects of non-stationary airfoil theory and its practical application. J. Aeronaut. Sci. 8, 104–108 (1941). [MF 5328]

In the first part of the paper the author gives a summary of results obtained by von Kármán and Sears [J. Aeronaut. Sci. 5, 379–390 (1938)] for a rigid airfoil of infinite span in "translatory" and "rotational" oscillatory motion while the airfoil itself is moving with a constant velocity through a fluid medium. Besides the usual assumptions that the fluid is incompressible and nonviscous, and that the flow satisfies the Kutta-Joukowski condition, it is further assumed that the amplitude of oscillation is small. This last assumption is used to linearize the problem. In the second part of the paper results of analysis are given for a rigid airfoil of infinite span moving with a constant horizontal velocity into a fluid medium which has a series of alternating up and down gusts, sinusoidally distributed in the horizontal direction. The lift is shown to be always acting at the quarter-chord point of the airfoil. In the third part of the paper the results stated in the first and second parts are used to calculate approximately the torsional oscillation of a fan blade operating behind a set of pre-rotation vanes. The paper concludes with a numerical example. *H. S. Tsien* (Pasadena, Calif.).

Pekeris, C. L. On the statistical theory of turbulence. J. Aeronaut. Sci. 8, 475–476 (1941). [MF 5474]

The author considers first a new method of determining the distribution function $P(u)$ of the turbulent velocities. He shows that its even and odd parts can be calculated if the mean values with respect to time of $\cos[zu(t)]$ and $\sin[zu(t)]$ are measured for all values of z. In the second part of the paper he develops a new method of determining the auto-correlation coefficient $R(\tau)$ of $u(t)$, that is, the correlation between the values of u at times t and $t+\tau$ at a point. A function $U(t, \theta)$ is defined as the mean value of $u(t)$ in the interval $t-\theta$ to $t+\theta$. Then $r(\theta)$ is defined as $\overline{U^2(t,\theta)/u^2(t)}$ (the bars denote mean values with respect to time), and it is deduced that

$$R(2\theta) = \frac{1}{2}\frac{d^2}{d\theta^2}[\theta^2 r(\theta)].$$

W. R. Sears (Inglewood, Calif.).

Theory of Elasticity

Neronoff, N. Sopra un moto verticale di un grave sospeso ad un filo elastico. Boll. Un. Mat. Ital. (2) 2, 420–423 (1940). [MF 5559]

Neronoff, N. P. Sulle vibrazioni elastiche longitudinali di una corda avvolgentesi, di lunghezza variabile. Boll. Un. Mat. Ital. (2) 2, 424–434 (1940). [MF 5560]

A rope is attached to a pulley and carries a mass m at one end O. The pulley is rotated with a known tangential velocity $v(t)$, which causes a vertical motion of m. Hooke's law is assumed for the tension T of the rope: $T=ku_x(x, t)$, where x is the distance from O of a point P of the rope in unstressed state, $u(x, t)$ is the elastic elongation of OP and $u_x \equiv \partial u/\partial x$. The sliding of the rope on the pulley is neglected. In the first paper a uniform distribution of u_x along a massless rope is considered. The displacement of m satisfies a third order differential equation, which is integrated approximately by means of the Bessel and of the elementary transcendental functions when $v(t)=a+bt$.

In the second paper the mass of the rope is taken into account, and its linear density ρ is assumed constant. Then: $u_{tt}(x, t)=\rho^{-1}ku_{xx}(x, t)-m^{-1}ku_x(0, t)$. The initial conditions for $u(x, t)$ are: $u(x, 0)=f(x)$, $u_t(x, 0)=\varphi(x)$, where f, φ and f_x are continuous functions of x. The boundary conditions in the case of an upward motion are: $u(0, t)=0$, $\rho^{-1}ku_{xx}(L, t)+L_t(t)u_{xt}(L, t)=g-v_t(t)$, where $L\equiv L(t)$ is the length of the vertical part of the rope when it is stressed. (Similar boundary conditions hold also for a downward motion.) $L(t)\approx L(0)+\int_0^t v(t)dt$ if $1+u_x(x, t)\approx 1$. Introducing this approximation an explicit expression of $u(x, t)$ is obtained.

I. Opatowski (Minneapolis, Minn.).

Platrier, Charles. Sur l'intégration des équations indéfinies de l'équilibre élastique. C. R. Acad. Sci. Paris 212, 749–751 (1941). [MF 4931]

The general solution of the equations of elastic equilibrium is expressed in terms of harmonic functions and of a function the Laplacian of which is the cubical dilation.

A. Weinstein (Toronto, Ont.).

Tolotti, Carlo. Applicazione di un nuovo metodo di M. Picone all'integrazione delle equazioni dell'elasticità in un parallelepipedo rettangolo. Atti del secondo Congresso dell'Unione Matematica Italiana, Bologna, 4–6 Aprile 1940, pp. 422–424. [MF 5763]

Vecoua, Elias. Application of a method of the Academician N. Muschelishvili to the solution of boundary value problems of the plane theory of elasticity of anisotropic media. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 719–724 (1940). (Russian) [MF 5289]

The determination of the state of stress in a two-dimensional anisotropic medium contained within a region R (whose boundary C is subjected to a specified distribution of stress) is known to reduce to the search for a pair of analytic functions $\varphi(z)$ and $\psi(z_1)$ defined respectively in the regions R and R_1, where R lies in the plane of the complex variable $z=x+iy$, and the region R_1 is obtainable from R by an affine transformation defined by $z_1=x+iy/k$ ($k\neq 1$, real). The equation for the determination of $\varphi(z)$ and $\psi(z_1)$ is

$$(1+k)\varphi(t)+(1-k)\overline{\varphi(t)}+2\psi(t_1)=f_1(t)-ikf_2(t),$$

where $t=\xi+i\eta$ is a variable point on the boundary C, $t_1=\xi+i\eta/k$ is the corresponding point on the boundary C_1 of R_1, and $f_1(t)$ and $f_2(t)$ are prescribed real functions on the boundary C. It is shown that, if the region R is finite and simply-connected, and $f_1(t)$ and $f_2(t)$ satisfy Hölder's condition, then the functions φ and ψ can be determined from the solution of a certain Fredholm integral equation. The method of reducing the problem to the solution of Fredholm's equation is that employed by N. Muschelishvili in the corresponding isotropic problem [C. R. (Doklady) Acad. Sci. URSS (N.S.) **3**, 7–11, 73–77 (1934)].

I. S. Sokolnikoff (Madison, Wis.).

Sawin, G. Der Druck eines Systems absolut starrer Profile auf eine anisotrope Halbebene. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 725–730 (1940). (Russian. German summary) [MF 5290]

The author investigates the state of stress in an anisotropic elastic half-plane whose boundary ($y=0$) is subjected to pressures by several rigid bodies. The line of contact of the body S_k with the half-plane is of the form $y=f_k(x)$, $a_k\leqq x\leqq b_k$, $k=1, 2, \cdots, n$. Two cases are considered: (a) each body S_k is subjected to a vertical force P_k pressing the body to the half-plane; (b) the bodies S_k are rigidly connected and a force of magnitude P is applied to the entire system. A similar problem for an isotropic half-plane was considered by Begiaschvili [C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 914–916 (1940); these Rev. **2**, 332].

I. S. Sokolnikoff (Madison, Wis.).

Locatelli, Piero. Sullo stato di tensione elastica nei continui omogenei. Pont. Acad. Sci. Acta **5**, 21–33 (1941). [MF 4881]

The author derives three differentially independent nonlinear differential equations which are satisfied by any stress tensor of a homogeneous isotropic body. Tensor calculus is used throughout the paper. *A. Weinstein.*

Steuermann, E. On the question of local deformations in elastic bodies pressed against one another. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 738–741 (1941). [MF 5241]

The author considers a special problem arising from the classical theory of Hertz for elastic bodies in contact [compare the author's generalizations of the Hertz theory in C. R. Acad. Sci. URSS (N.S.) **29**, 179–181, 182–184 (1940); cf. these Rev. **2**, 332]. The special question treated is that of the determination of the domain of contact of the solids pressed together. Only in special cases, such as that of a rigid stamp pressed into a body, is the domain of contact known in advance. The author shows, however, that the domain is uniquely determined for a large class of problems through the condition that the pressure between the bodies be finite. *J. J. Stoker* (New York, N. Y.).

Deuker, Ernst-August. Beitrag zur Theorie endlicher Verformungen und zur Stabilitätstheorie des elastischen Körpers. Deutsche Math. **5**, 546–562 (1941). [MF 4803]

The author develops the theory of finite deformations, using comoving or Lagrangian coordinates. The stress-strain relations are derived from an energy principle, in which, however, the heat flow for an isothermal change is not discussed. Although the author refers to the earlier work of L. Brillouin [Ann. Physique **3**, 251 (1925)] and F. D. Murnaghan [Amer. J. Math. **59**, 235–260 (1937)], he does not compare his methods with those given in these papers. It appears to the reviewer that Murnaghan's paper contains, in more complete and general form, all the essentials of the paper under review, except some concluding remarks concerning stability. *J. L. Synge* (Toronto, Ont.).

Platrier, Charles. Symétrie de révolution des tensions dans un milieu homogène isotrope en équilibre élastique. C. R. Acad. Sci. Paris **212**, 680–682 (1941). [MF 4927]

The author discusses an example illustrating the fact that the assumption of symmetric stresses does not imply a symmetric distribution of displacements. *A. Weinstein.*

Lekhnitsky, S. G. Plane problem of the theory of elasticity for a medium with a slightly expressed aelotropy. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 845–848 (1941). [MF 5514]

In an earlier paper [the same C. R. **31**, 433–436 (1941); cf. these Rev. **3**, 29] the author indicated that the stress function F for a slightly aelotropic medium can be expressed by the expansion

$$F=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_1^m\lambda_2^n F_{mn},$$

where λ_1, λ_2 are parameters which are small compared to unity (expressing the departure of the medium from an isotropic medium) and F_{00} is the biharmonic stress function in the same isotropic medium. In the present paper the aelotropic medium is an infinite region containing a "curvilinear notch," which can be mapped upon the unit circle $|\zeta|=1$ by a rational function $z=\omega(\zeta)$. The forces on the boundary of the notch are self equilibrating and the stresses and displacements are bounded at infinity. In this case the problem is solved by determining two sequences of analytic functions $\phi_{mn}(\zeta)$ and $\psi_{mn}(\zeta)$ ($m, n=0, 1, 2, \cdots$), which are negative power series with coefficients determined in terms of the boundary conditions on the notch and at infinity. The method is applied to an infinite plate under a tensile stress in a given direction. The curvilinear notch is an approximate "square" hole which maps on $|\zeta|=1$ by $z=\omega(\zeta)=a(\zeta+b\zeta^{-3})$ for $a>0$, $|b|>\frac{1}{3}$. The author shows that if $b=0$ his results reduce to known results for a circular hole in an infinite plate to the degree of accuracy which the approximation affords when only four of the terms F_{mn} are employed. *D. L. Holl* (Ames, Iowa).

Faedo, Sandro. Deformazione di una piastra a spessore variabile soggetta a pressione. Atti del secondo Congresso dell'Unione Matematica Italiana, Bologna, 4–6 Aprile 1940, pp. 205–209. [MF 5760]

Applying the theory of the circular elastic plate of vari-

able thickness the author determines the deformation of a spherical biconcave or biconvex lens supported by a circular ring and subjected to a uniform pressure on one side.

W. Prager (Providence, R. I.).

Ôkubo, Hajimu. Stress systems in an aeolotropic rectangular plate. Z. Angew. Math. Mech. **21**, 162–176 (1941).

The author makes use of expansion methods to obtain formal expressions for stresses in a thin non-isotropic rectangular plate whose edges are subjected to an arbitrary distribution of tractions. The plate is assumed to be in a state of generalized plane stress, and the stress function $\chi(x, y)$ is taken in the form $\chi=\Re[f_1(x+iky)+f_2(x+iy)]$, where $k^2=E_1/E_2=\nu_2/\nu_1$ and the E_j and ν_j are the Young's moduli and Poisson's ratios associated with the x- and y-directions. The case of a concentrated force acting in the plane of the plate, when its edges are free from traction, is also considered.

I. S. Sokolnikoff (Madison, Wis.).

Reissner, Eric. Least work solutions of shear lag problems. J. Aeronaut. Sci. **8**, 284–291 (1941). [MF 5039]

The distribution of stress in the cover sheets of box-beams under bending loads is found by assuming parabolic distributions of bending stress across the sheets and determining the variation along the span of the shape of the parabolas by the use of the method of least work. This leads to a second order differential equation for the variation along the span of the curvature at the vertices of the stress parabolas. This equation can be integrated in terms of elementary functions in certain cases. An explicit solution is given for beams of constant cross-section under different distributions of load.

H. W. March (Madison, Wis.).

Winter, George. Lateral stability of unsymmetrical I-beams and trusses in bending. Proc. Amer. Soc. Civil Engrs. **67**, 1851–1864 (1941). [MF 5826]

When small loads are applied in the longitudinal vertical plane of symmetry, a beam is deflected in the vertical plane only and this form of bending is a stable one. By increasing the external loads a limiting value is reached at which the plane form of bending is unstable and lateral bending accompanied by rotation ensues. To determine the critical load the author assumes for the rotation an approximation of a sine wave which satisfies the end conditions and applies the energy method of equating the sum of the strain energy of lateral bending and of rotation to the change of the potential energy of the external loads. In applying this condition for the determination of the critical load it is assumed that the flexural strain energy of bending in the vertical plane remains unchanged during lateral buckling and that the deformations of all transverse sections of the beam are adequately described by the displacement of the flexural center of the beam. Non-symmetrical I-beams and beams with rectangular sections are considered under the action of a single central load or of a moment applied at ends of the beam.

D. L. Holl (Ames, Iowa).

Ruchadze, A. K. Problem of bending of beams near to prismatical ones. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 577–582 (1940). (Russian) [MF 5284]

The author considers the problem of bending a nearly prismatic bar by a transverse force applied at the end of the bar. The approach to the problem is that used by D. B. Panov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **20** (1938)] in a study of torsion of rods whose lateral surface has the form $f[x(1-kz), y(1-kz)]=0$, k being a small parameter. The coordinates ξ, η, ζ, defined by $\xi=x(1-kz)$, $\eta=y(1-kz)$, $\zeta=z$, are introduced and the problem is reduced to a determination of a system of stresses satisfying in the space of the variables ξ, η, ζ the usual equations of equilibrium and certain boundary conditions on the surface $f(\xi, \eta)=0$. The system of stresses satisfying these conditions does not yield a distribution of forces over the ends of the bar that is statistically equivalent to a transverse force. Thus it is necessary to superimpose a solution of a certain problem of St. Venant on the solution given in the paper.

I. S. Sokolnikoff (Madison, Wis.).

Ruchadze, A. K. On the problem of deformation of a beam with slightly curved axis. Mitt. Akad. Wiss. Georgischen SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 35–42 (1941). (Russian) [MF 5296]

The problem of bending a slightly curved bar by a transverse force applied at the end of the bar is treated in a way parallel to that of the paper reviewed above. The equation of the lateral surface of the bar is taken in the form $f(x+kz^2, y)=0$, where the parameter k is so small that its powers higher than the first can be neglected. P. M. Riz considered the problem of torsion and bending of such bars by couples applied at the ends [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 110–113, 229–232 (1939); these Rev. **2**, 176].

I. S. Sokolnikoff (Madison, Wis.).

Panetti, Modesto. Comportamento elastico delle molle ad elica cilindrica per flessione dell'asse. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 541–547 (1940). [MF 4786]

Determination of a critical velocity in a certain engineering problem.

A. Weinstein (Toronto, Ont.).

Sadowsky, Michael A. Equiareal pattern of stress trajectories in plane plastic strain. J. Appl. Mech. **8**, A-74–A-76 (1941). [MF 5814]

An orthogonal curvilinear system of coordinates u, v is called equiareal if two sets of values u_i, v_i can be selected so as to render equal in area the finite curvilinear rectangles formed by the parameter curves $u=u_i$, $u=u_{i+1}$, $v=v_i$, $v=v_{i+1}$. With the linear element $ds^2=Edu^2+Gdv^2$, the condition for an equiareal orthogonal system is $\partial^2 \log EG/\partial u\partial v=0$. It is proved that the stress trajectories in plane plastic strain form an equiareal system.

W. Prager (Providence, R. I.).

Sadowsky, M. A. An extension of the sand-heap analogy in plastic torsion applicable to cross sections having one or more holes. J. Appl. Mech. **8**, A-166–A-168 (1941). [MF 5815]

A. Nádai's sandheap analogy in plastic torsion is extended to cross sections with holes. The plastic stress function has a gradient of constant magnitude, vanishes along the outer boundary of the cross section and assumes a suitable constant value along the boundary of each hole. This function can be obtained experimentally by heaping sand on a horizontal tray which has the shape of the cross section. Tubes fitting the holes are adjusted vertically in such a manner that their horizontal brims are just completely submerged in the sand.

W. Prager (Providence, R. I.).